Heat Flow and Perimeter in $\mathbb{R}^m$

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Abstract Let $\Omega$ be an open set in Euclidean space $\mathbb{R}^m$ with finite perimeter $\mathcal{P}(\Omega)$, and with $m$-dimensional Lebesgue measure $|\Omega|$. It was shown by M. Preunkert that if $T(t)$ is the heat semigroup on $L^2(\mathbb{R}^m)$ then $H_\Omega(t) := \int_\Omega T(t) 1_\Omega(x) dx = |\Omega| - \pi^{-1/2} \mathcal{P}(\Omega)t^{1/2} + o(t^{1/2}), \ t \downarrow 0$. $H_\Omega(t)$ represents the amount of heat in $\Omega$ if $\Omega$ is at initial temperature 1 and if $\mathbb{R}^m \setminus \Omega$ is at initial temperature 0. In this paper we will compare the quantitative behaviour of $H_\Omega(t)$ with the usual heat content $Q_\Omega(t)$ associated to the Dirichlet heat semigroup on $\Omega$. We analyse the heat content for horn-shaped open sets of the form $\Omega(\alpha, \Sigma) = \{ (x, x') \in \mathbb{R}^m : x' \in (1 + x)^{-\alpha} \Sigma, x > 0 \}$, where $\alpha > 0$, and where $\Sigma$ is an open set in $\mathbb{R}^{m-1}$ with finite perimeter in $\mathbb{R}^{m-1}$, which is star-shaped with respect to 0. For $m \geq 3$ we find that there are four regimes with very different behaviour depending on $\alpha$, and a further two limiting cases where logarithmic corrections appear.

Keywords Heat flow · Perimeter · Euclidean space

Mathematics Subject Classification (2010) 35K05

1 Introduction

In this paper we will obtain some results for the heat flow from sets in Euclidean space $\mathbb{R}^m$ into their complement. Let

$$p_m(x, y; t) = (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)},$$

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and let $\Omega$ be a non-empty open set in $\mathbb{R}^m$. Let $u_\Omega$ be the unique weak solution of
\[
\Delta u = \frac{\partial u}{\partial t}, \quad x \in \mathbb{R}^m, \ t > 0,
\]
with initial condition
\[
u(x; 0) = 1_\Omega(x), \ x \in \mathbb{R}^m,
\]
where $1_\Omega : \mathbb{R}^m \mapsto \{0, 1\}$ is the characteristic function of $\Omega$. We have that
\[
u_\Omega(x; t) = \int_\Omega dyp_m(x, y; t).
\]

We define the heat content of $\Omega$ in $\mathbb{R}^m$ at time $t$ by
\[
H_\Omega(t) = \int_\Omega dx u_\Omega(x; t) = \int_\Omega dyp_m(x, y; t).
\]

It was shown [8–10] that if $\Omega$ is an open set in $\mathbb{R}^m$ with finite Lebesgue measure $|\Omega|$ and with finite perimeter $P(\Omega)$ then
\[
P(\Omega) = \lim_{t \to 0} \left( \frac{\pi}{t} \right)^{1/2} \int_{\Omega \times (\mathbb{R}^m \setminus \Omega)} dx dyp_m(x, y; t).
\]

Since
\[
\int_{\Omega \times (\mathbb{R}^m \setminus \Omega)} dx dyp_m(x, y; t) = |\Omega| - H_\Omega(t),
\]
we conclude the following.

**Theorem 1** If $\Omega$ is an open set in $\mathbb{R}^m$ with finite Lebesgue measure and with finite perimeter then
\[
H_\Omega(t) = |\Omega| - \pi^{-1/2} P(\Omega) t^{1/2} + o(t^{1/2}), \ t \downarrow 0.
\]

Let $v_\Omega$ be the unique weak solution of
\[
\Delta v = \frac{\partial v}{\partial t}, \quad x \in \Omega, \ t > 0,
\]
with initial condition
\[
v(x; 0) = 1, \ x \in \Omega,
\]
and with Dirichlet boundary condition
\[
v(x; t) = 0, \ x \in \partial \Omega, \ t > 0.
\]

The heat content of $\Omega$ at time $t$ is defined by
\[
Q_\Omega(t) = \int_\Omega dx v_\Omega(x; t).
\]

The latter quantity has been studied extensively in the general setting of open bounded sets with smooth boundaries in complete Riemannian manifolds. See for example [4, 6]. We recall the following definition in [3].