Heat Content and Inradius for Regions with a Brownian Boundary

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Abstract In this paper we consider \( \beta[0, s] \), Brownian motion of time length \( s > 0 \), in \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) and on the \( m \)-dimensional torus \( T^m \). We compute the expectation of (i) the heat content at time \( t \) of \( \mathbb{R}^m \setminus \beta[0, s] \) for fixed \( s \) and \( m = 2, 3 \) in the limit \( t \downarrow 0 \), when \( \beta[0, s] \) is kept at temperature 1 for all \( t > 0 \) and \( \mathbb{R}^m \setminus \beta[0, s] \) has initial temperature 0, and (ii) the inradius of \( T^m \setminus \beta[0, s] \) for \( m = 2, 3, \ldots \) in the limit \( s \to \infty \).

Keywords Laplacian · Brownian motion · Wiener sausage · Heat content · Inradius · Spectrum

Mathematics Subject Classifications (2010) 35J20 · 60G50

1 Introduction and Main Results

Asymptotic properties of the heat content and the inradius for regions with a fractal boundary have received a lot of attention in the literature. Most of the focus has been on porous regions (e.g. the \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) from which a Poisson cloud of non-polar sets is removed [4, 20]), and regions with a fractal polygonal boundary e.g. the von Koch snow flake and its relatives [2, 3, 6]. In this paper we consider the region obtained from \( \mathbb{R}^m \) or the \( m \)-dimensional torus \( T^m \) by cutting out a Brownian path of time length \( s \). In Sections 1.1 and 1.2 we consider the
heat content, and in Section 1.3 the inradius. We formulate some open problems in Section 1.4. The proofs are deferred to Sections 2–3.

1.1 Heat Content Outside Compact Sets

Let $K$ be a compact non-polar set in $\mathbb{R}^m$ with boundary $\partial K$, and let $v : \mathbb{R}^m \setminus K \times [0, \infty) \to \mathbb{R}$ be the unique weak solution of the heat equation

$$\begin{aligned}
\Delta v(x; t) &= \partial_v(x; t)/\partial t, \quad x \in \mathbb{R}^m \setminus K, \ t > 0, \\
v(x; t) &= 1, \quad x \in \partial K, \ t > 0, \\
v(x; 0) &= 0, \quad x \in \mathbb{R}^m \setminus K.
\end{aligned} \tag{1.1}$$

Then $v(x; t)$ represents the temperature at point $x$ at time $t$ when $\partial K$ is kept at temperature 1 and the initial temperature is 0. The heat content of $\mathbb{R}^m \setminus K$ at time $t$ is defined by

$$E_K(t) = \int_{\mathbb{R}^m \setminus K} v(x; t) \, dx. \tag{1.2}$$

If $\partial K$ is $C^\infty$, then $E_K(t)$ has an asymptotic series expansion for $t \downarrow 0$ of the form

$$E_K(t) = \sum_{j=1}^J a_j(K) t^{j/2} + O(t^{(J+1)/2}), \quad t \downarrow 0, \ J \in \mathbb{N}, \tag{1.3}$$

where the coefficients are local geometric invariants of $\mathbb{R}^m \setminus K$. In particular,

$$\begin{aligned}
a_1(K) &= 2\pi^{-1/2} \int_{\partial K} \, dz, \\
a_2(K) &= 2^{-1}(m - 1) \int_{\partial K} H(z) \, dz,
\end{aligned}$$

where $dz$ is the surface measure on $\partial K$, and $H(z)$ is the mean curvature at $z$ of $\partial K$ with inward orientation. Formulas of this type can be found in the general setting of Riemannian manifolds and Laplace-type operators [5, 11]. The case where $\partial K$ is only $C^3$ was settled by probabilistic tools in [7], and the expansion in Eq. 1.3 holds for $J = 2$.

The asymptotic behaviour for $t \to \infty$ is different and its analysis does not require smoothness of $\partial K$. For $m = 3$ it is shown in [12, 16] (see [18] for earlier results) that if $K$ is a compact set, then

$$E_K(t) = \sum_{j=1}^3 b_j(K) t^{(j-3)/2} + O(t^{-1/2}), \quad t \to \infty, \tag{1.4}$$

with

$$\begin{aligned}
b_1(K) &= \text{cap}(K), \\
b_2(K) &= 2^{-1}\pi^{-3/2}\text{cap}(K)^2, \\
b_3(K) &= (4\pi)^{-2}\text{cap}(K)^3 - |K| - (8\pi)^{-1} \int_K \int_K \|x - y\| \mu_K(dx)\mu_K(dy), \tag{1.5}
\end{aligned}$$