Integrals Along Rough Paths via Fractional Calculus

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Abstract
Using fractional calculus, we introduce an integral along $\beta$-Hölder rough paths for any $\beta \in (0, 1]$. This is a natural generalization of the Riemann–Stieltjes integral along smooth curves. We prove that, under suitable conditions on the integrand, this integral is a continuous functional with respect to the Hölder topology. As a result, this provides an alternative definition of the first level path of the rough integral along geometric Hölder rough paths.

Keywords
Fractional derivatives · Stieltjes integrals · Rough paths

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1 Introduction
The theory of rough paths introduced by Lyons [6] has produced a framework of multidimensional controlled differential equations driven by non-smooth functions, known as rough differential equations. These differential equations have led to useful methods for studying stochastic calculus; in particular, it enables us to take a pathwise approach to classical stochastic calculus and provides a convenient tool for the study of a large class of stochastic processes that are not semimartingales, such as fractional Brownian motions.

Hu and Nualart introduced an alternative approach to the theory of rough paths based on fractional calculus [4]. They defined integrals along Hölder continuous functions of order $\beta \in (1/3, 1/2)$ by combining the concepts of rough path analysis and the explicit expressions of the Riemann–Stieltjes integral in terms of fractional derivatives introduced by Zähle [12]. This integral provides another tool to study multidimensional controlled differential equations driven by Hölder continuous functions. For example, Besalú and Nualart [1]
made a study of stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H \in (1/3, 1/2)$. This approach is beneficial in that the integrals are described by using the usual Lebesgue integrals without approximation by Riemann sums, unlike the original definition of the rough integral. Further development in this direction could provide sophisticated access to the fundamental theory of rough paths.

The next natural question to consider is whether this approach is valid for less regular functions, that is, whether the rough integral can be expressed on the basis of fractional derivatives for $\beta$-Hölder continuous functions with $\beta$ less than $1/3$. In such cases, the situation becomes much more involved since we have to consider rough paths up to the $\lfloor 1/\beta \rfloor$-th level path. The purpose of this paper is to show that this is possible by generalizing the preceding studies [4, 12]. More precisely, we provide a definition of the integral along $\beta$-Hölder rough paths for any $\beta \in (0, 1]$ (Definition 2.3), which is explicitly described by fractional derivatives. This is an entirely new definition. To ensure the definition is reasonable, we prove that the integral is consistent with the Riemann–Stieltjes integral along smooth curves (Theorem 2.5) and is a continuous functional with respect to the $\beta$-Hölder distance (Theorem 2.6). As a result, our integral coincides with the first level path of the rough integral along geometric $\beta$-Hölder rough paths (Theorem 2.7).

One of the key ingredients for the definition of our integral is the integration by parts of fractional orders, as described by Hu and Nualart [4, Theorem 3.3]. Due to the lower regularity of these functions, the integrand has to be decomposed into the regular part and the remainder part. The latter is then replaced by the higher level path of the rough path for the integration to make sense. In this procedure, we have to take care of additional terms that successively arise from some integration by parts formulas of fractional orders and multiplicative property of the rough path. For this reason, the resulting formula involves some complicated terms. Once we have explicit expressions for the integral, however, it is not difficult to provide quantitative estimates for proving the continuity of the integration operator.

In the forthcoming paper [5], the definition of integration in this paper is adopted as integration of weakly controlled paths to provide an alternative proof of Lyons’ extension theorem for geometric Hölder rough paths together with an explicit expression of the extension map.

The remainder of this paper is organized as follows. In Section 2, we state the main theorems in addition to the definition of the integral. Some components of rough path analysis and fractional calculus are also provided. Sections 3 and 4 are devoted to the proofs of the main theorems.

## 2 Definition of the Integral and Main Theorems

In this section, we introduce the definition of the integral as well as the main theorems. We also present some components of rough path analysis and fractional calculus, following standard treatments for rough path analysis [2, 6–8] and for fractional calculus [10, 12].

### 2.1 Notation

Throughout this paper, $C$ denotes a positive constant, which may change line-by-line. Let $V$ and $W$ be finite-dimensional normed spaces with norms $\| \cdot \|_V$ and $\| \cdot \|_W$, respectively. Although the fundamental theory of rough paths is valid for suitable infinite-dimensional Banach spaces, we consider only finite-dimensional cases in this paper to avoid the technical