Algebraic Convergence Rate for Reflecting Diffusion Processes on Manifolds with Boundary

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Abstract A criteria for the algebraic convergence rate of diffusion semigroups on manifolds with respect to some Lipschitz norms in $L^2$-sense is presented by using a Lyapunov condition. As application, we apply it to some diffusion processes with heavy tailed invariant distributions. This result is further extended to the reflecting diffusion processes on manifolds with non-convex boundary by using a conformal change of the metric.

Keywords Algebraic convergence · Lyapunov condition · Lipschitz norm · Coupling · Non-convex boundary

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1 Introduction

In the past decades, great efforts have been made to the ergodic theory for Markov processes. One of the most popular topics in this direction is to investigate the convergence rate for the processes (see e.g. [8, 14, 19] and the references within). However, the work on algebraic (or polynomial) convergence, especially for diffusion processes on manifolds, is still limited; readers are referred to [9, 13, 14, 18, 22] for the background and the present status of this topic. This paper is devoted to studying the algebraic $L^2$-convergence for diffusion processes on Riemannian manifolds with boundary.

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Let \((M, g_M)\) be an \(n\)-dimensional complete connected Riemannian manifold possibly with convex boundary \(\partial M\). Let \(\nabla\) and \(\Delta\) be, respectively, the Levi-Civita connection and the Laplace-Beltrami operator. Define the second fundamental form by

\[
\Pi_g(X, Y) = -\langle \nabla_X N, Y \rangle, \quad \text{for all} \ X, Y \in T\partial M,
\]
where \(N\) is the inward unit normal vector field of the boundary and \(\langle \cdot, \cdot \rangle := g_M\). It is well known that the case \(\Pi_g \geq 0\) corresponds to the convexity of the boundary \(\partial M\).

Consider the elliptic operator

\[
L := \Delta + \nabla h,
\]
where \(h \in C^2(M)\) satisfies \(Z := \int_M e^{h(x)}dx < \infty\) and \(dx\) is the Riemannian volume measure. Suppose that \((X_t)\) is a (reflecting if \(\partial M \neq \emptyset\)) \(L\)-diffusion process with reversible measure \(\pi(dx) := Z^{-1}e^{h(x)}dx\). Throughout this article, we always assume that \((X_t)\) is non-explosive. This assumption immediately implies that there is a strongly continuous semigroup \(P_t\) on \(L^2(\pi)\) corresponding to \((X_t)\). The processes \((X_t)\) is said to have the algebraic convergence rate in \(L^2\)-sense if there exists a functional \(\Phi : L^2(\pi) \rightarrow [0, \infty]\), some constants \(q > 1\) and \(c > 0\) such that for all \(t > 0\) and \(f \in L^2(\pi)\),

\[
\|P_tf - \pi(f)\|^2 \leq ct^{1-q} \Phi(f),
\]
where \(\|\cdot\|^2\) denotes the \(L^2(\pi)\)-norm and \(\pi(f) := \int_M f d\pi\).

Let

\[
\text{Lip}(d) := \left\{ f \in L^2(\pi) : \|f\|_{\text{Lip}(d)} := \sup_{(x, y) \in (M \times M) \setminus (C \cup D)} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right\},
\]
where \(C := \{(x, y) : y \text{ is a cut-point of } x\}\), \(D := \{(x, x) : x \in M\}\) and \(d\) is a distance on \(M\). Here and thereafter, we set \(\|f\|^2_{\text{Lip}(d)} = \infty\) if \(f \in L^2(\pi) \setminus \text{Lip}(d)\) for simplicity. In this article, we investigate the algebraic convergence behavior of \(P_t\) with respect to the Lipschitz norm \(\|\cdot\|_{\text{Lip}(d)}\). Recall that if \(d\) is the Riemannian distance \(\rho\) on \(M\), then \(\|f\|_{\text{Lip}(d)} = \|\nabla f\|_\infty\) for \(f \in C^1(M) \cap \text{Lip}(d)\); if

\[
d(x, y) = \delta(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y, \end{cases}
\]
then \(\|f\|_{\text{Lip}(d)} = \text{Osc}_\rho(f) := \text{ess sup}_M(f) - \text{ess inf}_M(f)\). Moreover, we consider a special distance constructed under some curvature condition. More precisely, let \(\text{Ric}\) be the Ricci curvature tensor and \(\text{Hess}_h(X, Y) = \langle \nabla_X \nabla h, Y \rangle\) for \(X, Y \in TM\). We assume \((M, g_M)\) satisfy the following curvature condition.

**Assumption (H1)** \(\text{Ric} - \text{Hess}_h \geq -K\) for some \(K \geq 0\).

Recall that if \(K < 0\) in the above assumption, then \(P_t\) has exponential convergence rate (see e.g. [19, Chapter 2]). In this article, we deal with the case \(K \geq 0\) with the aim to obtain some criteria for the algebraic convergence rate with respect to \(\|\cdot\|^2_{\text{Lip}(\rho))}\), where for all \(r \in [0, \infty)\),

\[
F(r) := \int_0^r e^{-\frac{K}{\delta}x^2}dx.
\]

Before moving on, let us give some comments on this Lipschitz norm.

1. Since \(F(0) = 0\), \(F'(r) \geq 0\) and \(F''(r) \leq 0\) for \(r \in [0, \infty)\), we know that \(F \circ \rho\) is a distance over \(M\).
2. If \(K > 0\), then \(\int_0^\infty e^{-\frac{K}{\delta}x^2}dx < \infty\). Thus we have \(F(r) \sim O(1)\) as \(r \rightarrow \infty\); and \(F(r) \sim O(r)\) as \(r \rightarrow 0\). If \(K = 0\), then \(F(r) = r\).