Quaternary universal forms over $\mathbb{Q}(\sqrt{13})$

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Abstract Let $F = \mathbb{Q}(\sqrt{m})$ be a real quadratic field over $\mathbb{Q}$ with $m$ a square-free positive rational integer and $O$ be the integer ring in $F$. A totally positive definite integral $n$-ary quadratic form $f = f(x_1, \ldots, x_n) = \sum_{1 \leq i, j \leq n} \alpha_{ij} x_i x_j$ ($\alpha_{ij} = \alpha_{ji} \in O$) is called universal if $f$ represents all totally positive integers in $O$. Chan, Kim and Raghavan proved that ternary universal forms over $F$ exist if and only if $m = 2, 3, 5$ and determined all such forms. There exists no ternary universal form over real quadratic fields whose discriminants are greater than 12.

In this paper we prove that there are only two quaternary universal forms (up to equivalence) over $\mathbb{Q}(\sqrt{13})$. For the proof of universality we apply the theory of quadratic lattices.

Keywords Universal forms · Quadratic lattices · Real quadratic fields

Mathematics Subject Classification (2000) Primary 11E25 · Secondary 11E20

1 Introduction

Let $F = \mathbb{Q}(\sqrt{m})$ be a real quadratic field over $\mathbb{Q}$ with $m$ a square-free positive rational integer and $O$ be the integer ring in $F$. We also denote by $O^+$ the set of all totally positive integers in $O$. A totally positive definite integral $n$-ary quadratic form $f = f(x_1, \ldots, x_n) = \sum_{1 \leq i, j \leq n} \alpha_{ij} x_i x_j$ ($\alpha_{ij} = \alpha_{ji} \in O$) is called universal if $f$ represents all integers in $O^+$. Chan, Kim and Raghavan [1] proved that ternary universal forms over $F$ exist if and only if $m = 2, 3, 5$ and determined all such forms. The question occurs whether there exist quaternary universal forms over real quadratic fields whose discriminants are greater than 12.
In this paper we prove that there are only two quaternary universal forms over \( \mathbb{Q}(\sqrt{13}) \), up to equivalence. Those forms are:

\[
\begin{align*}
    x_1^2 + x_2^2 + \frac{5 + \sqrt{13}}{2} x_3^2 + \frac{5 - \sqrt{13}}{2} x_4^2 + 2x_3x_4, \\
    x_1^2 + 2x_2^2 + \frac{5 + \sqrt{13}}{2} x_3^2 + \frac{5 - \sqrt{13}}{2} x_4^2 + 2x_2x_3 + 2x_2x_4.
\end{align*}
\]

For the proof of universality we apply the theory of quadratic lattices. Let \((V, Q)\) be a totally positive definite quadratic space over \( F \) and \( B \) a bilinear form associated with the quadratic form \( Q \). Let \( L \) be an \( \mathcal{O} \)-lattice on \( V \) (we simply call “lattice” in the following), that is, a finitely generated \( \mathcal{O} \)-module in \( V \) which contains a basis of \( V \) over \( F \). We assume that all lattices are integral, that is, the set of all values of bilinear form \( B \) is contained in \( \mathcal{O} \).

For a lattice \( L \) we denote by \( \text{cls}(L) \) and \( \text{gen}(L) \) the class and the genus of \( L \), respectively. Suppose \( R = \mathcal{O} \) or its localization \( \mathcal{O}_p \) at a prime spot \( p \) in \( F \). Let \( L \) be \( \mathcal{O} \)-lattices. We say \( L \) represents \( l \) (over \( R \)) if there is an injective linear map \( \sigma : l \rightarrow L \) such that \( Q(\sigma(x)) = Q(x) \) for all \( x \in l \) and write \( l \hookrightarrow L \). For \( \alpha \in R \) we say \( \alpha \) is represented by \( L \) if \( \langle \alpha \rangle \hookrightarrow L \). We call an \( \mathcal{O} \)-lattice \( L \) universal if \( L \) represents all \( \alpha \in \mathcal{O}^+ \). For \( \alpha, \beta \in R \) we denote by \( \alpha \sim \beta \) in \( R \) if \( \alpha = \beta u^2 \) for some unit \( u \) in \( R \). Put \( I_n := \perp_n \langle 1 \rangle \). For \( \alpha \in \mathcal{O} \) we put \( I_n(\alpha) := I_n \perp \langle \alpha \rangle \). See [4] for more general terminologies and notations on quadratic lattices.

We fix \( F = \mathbb{Q}(\sqrt{13}) \) in the following. We put \( \omega = (1 + \sqrt{13})/2 \) and \( \epsilon = 1 + \omega \) the fundamental unit in \( F \). Note that \( \omega^2 = \omega + 3 \). We also note \( \mathcal{O} = \mathbb{Z}[1, w] \). For \( \alpha \in F \) we denote by \( \overline{\alpha} \) the conjugate of \( \alpha \). We also note that the ideals \( (2) \), \( (\omega) \), \( (\overline{\omega}) \) are all prime ideals in \( F \) and \( (\omega) \neq (\overline{\omega}) \), \( \omega \overline{\omega} = -3 \).

### 2 Several ternary lattices

In this section we refer to the property of several ternary lattices.

For a lattice \( M \) we define the mass of \( M \) by

\[
m(M) = \sum_{M_i} \frac{1}{|O(M_i)|},
\]

where \( M_i \) runs through representatives of all classes in the genus of \( M \) and \( |O(M)| \) the order of orthogonal group of \( M \).

We put

\[
E_2 := \left\langle \begin{pmatrix} 2 + \omega & 1 \\ 1 & 3 - \omega \end{pmatrix} \right\rangle, \quad E_3 := \left\langle \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 + \omega & 0 \\ 1 & 0 & 3 - \omega \end{pmatrix} \right\rangle,
\]

\[
G_2(\rho) := \left\langle \begin{pmatrix} 2 & 1 \\ 1 & \rho \end{pmatrix} \right\rangle, \quad G_3(\rho) := \left\langle \begin{pmatrix} \overline{\rho} & 1 & 0 \\ 1 & \rho & 1 \\ 0 & 1 & \overline{\rho} \end{pmatrix} \right\rangle,
\]

where \( \rho = 2 + \omega \) or \( 3 - \omega \).