Ramanujan summation and the exponential generating function $\sum_{k=0}^{\infty} \frac{x^k}{k!} \zeta'(-k)$

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Received: 21 January 2009 / Accepted: 31 March 2009 / Published online: 2 December 2009
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Abstract In the sixth chapter of his notebooks, Ramanujan introduced a method of summing divergent series which assigns to the series the value of the associated Euler-MacLaurin constant that arises by applying the Euler-MacLaurin summation formula to the partial sums of the series. This method is now called the Ramanujan summation process. In this paper we calculate the Ramanujan sum of the exponential generating functions $\sum_{n \geq 1} \log n e^{nz}$ and $\sum_{n \geq 1} H_n^{(j)} e^{-nz}$ where $H_n^{(j)} = \sum_{m=1}^{n} \frac{1}{m^j}$. We find a surprising relation between the two sums when $j = 1$ from which follows a formula that connects the derivatives of the Riemann zeta-function at the negative integers to the Ramanujan sum of the divergent Euler sums $\sum_{n \geq 1} n^k H_n$, $k \geq 0$, where $H_n = H_n^{(1)}$. Further, we express our results on the Ramanujan summation in terms of the classical summation process called the Borel sum.

Keywords Divergent series · Euler sums · Generating function · Riemann zeta-function · Borel sum · Laplace transform

Mathematics Subject Classification (2000) 11M06 · 65B15 · 40G99
1 Introduction

Let
\[
\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.
\]
(1)

\(B_n(x)\) is the \(n\)th Bernoulli polynomial and \(B_n(0) = B_n\) is the \(n\)th Bernoulli number.

At the beginning of the sixth chapter of his Notebooks [3], Ramanujan writes the Euler–MacLaurin formula for the partial sums
\[
a(1) + a(2) + \cdots + a(x - 1) = C + \int_1^x a(t)dt + \sum_{k\geq 1} B_k \frac{a^{k-1}(x)}{k!}
\]
(2)
of an infinite series \(\sum_{n=1}^{\infty} a(n)\) and assigns the value of the constant ‘\(C\)’ the sum of the series. For example, one has
\[
\sum_{n=1}^{\infty} \frac{1}{n} = \gamma
\]
(3)
by this process, where \(\mathcal{R}\) denotes the Ramanujan summation and \(\gamma\) is the Euler-Mascheroni constant This follows from the formula [2]
\[
\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).
\]
(4)

It was pointed out by Hardy (Sect. 6, Chap. 13 of [13]) that this definition makes the Ramanujan sum of a series ambiguous as the sum depends on a particular parameter that appears as a lower limit of an integral. In [6], Candelpergher, Coppo and Delabaere formulated the Ramanujan summation in a rigorous manner so that the uniqueness of the sum of the series is established. They also proved several properties of the Ramanujan summation and summed many well known divergent and convergent series. One of the results they obtained using this summation process was that
\[
\sum_{n\geq 1}^{\mathcal{R}} H_n = \frac{3}{2} \gamma + \frac{1}{2} - \frac{1}{2} \log(2\pi),
\]
(5)
where \(H_n = \sum_{j=1}^{n} \frac{1}{j}\) is the \(n\)th harmonic number and \(H_0 = 0\). Note that
\[
\zeta'(0) = -\frac{1}{2} \log 2\pi.
\]
(6)

So the natural question to ask now is whether \(\sum_{n\geq 1}^{\mathcal{R}} n^k H_n\) is related to \(\zeta'(-k)\) for \(k \geq 1\).