On the 2- and 4-dissections of Ramanujan’s continued fraction and its reciprocal

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Abstract We present elementary proofs, using only Jacobi’s triple product identity, of four identities of Ramanujan and eight identities of Hirschhorn relating to the 2-dissection and the 4-dissection of Ramanujan’s continued fraction and its reciprocal, and of two identities from Ramanujan’s famous list of forty.

Keywords 2-dissection · 4-dissection · Ramanujan’s continued fraction · List of forty

Mathematics Subject Classification (2000) 11A55

1 Introduction

In the first volume on Ramanujan’s Lost Notebook [1], we find in Entries 4.3.1 and 4.3.2 product formulas for the four series obtained by 2-dissecting the series for the Rogers–Ramanujan continued fraction and its reciprocal. The authors give proofs based on two other identities, found in a list of forty identities stated by Ramanujan, and brought to public notice by Birch [4]. Proofs in the spirit of Ramanujan of almost all of these forty have been given in an eight-author paper [3], and are to be further discussed in Andrews and Berndt [2].

Some years ago I discovered product formulas for the eight series obtained by 4-dissecting the Rogers–Ramanujan continued fraction and its reciprocal and these were presented as conjectures in the paper in which I settled the 5-dissections [5]. These product formulas were subsequently proved by Lewis and Liu [7], who relied on a theorem concerning residues of functions of a complex variable satisfying a particular functional equation [6, Theorem 1].
I have since found elementary proofs of all 14 identities mentioned above, using only Jacobi’s triple product identity,

\[
\prod_{n \geq 1} (1 + aq^{2n-1})(1 + a^{-1}q^{2n-1})(1 - q^{2n}) = \sum_{-\infty}^{\infty} a^n q^{n^2},
\]

and the purpose of this note is to present these proofs.

In what follows, we adopt the by now standard notation

\[
(a; q)_\infty = \prod_{k \geq 0} (1 - aq^k),
\]

\[
(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty,
\]

\[
\left(\frac{a_1, a_2, \ldots, a_m}{b_1, b_2, \ldots, b_n}; q\right)_\infty = \frac{(a_1, a_2, \ldots, a_m; q)_\infty}{(b_1, b_2, \ldots, b_n; q)_\infty}.
\]

Let us start with Ramanujan’s four identities for the 2-dissection of his continued fraction and its reciprocal.

If

\[
C(q) = 1 + \frac{q}{1 + 1 + 1 + \cdots} = \left(\frac{q^2, q^3; q^5}{q^2, q^3; q^5}\right)_\infty = \sum_{n \geq 0} v_n q^n,
\]

\[
C(q)^{-1} = \frac{1}{1 + 1 + 1 + \cdots} = \left(\frac{q, q^4}{q^2, q^3; q^5}\right)_\infty = \sum_{n \geq 0} u_n q^n
\]

then

\[
\sum_{n \geq 0} v_{2n} q^n = \left(\frac{q^4, q^4, q^6, q^6, q^{10}}{q^3, q^5, q^5, q^7, q^{10}}\right)_\infty,
\]

\[
\sum_{n \geq 0} v_{2n+1} q^n = \left(\frac{q, q^4, q^6, q^9}{q^2, q^5, q^5, q^8, q^{10}}\right)_\infty,
\]

\[
\sum_{n \geq 0} u_{2n} q^n = \left(\frac{q^2, q^2, q^8, q^8}{q, q^5, q^5, q^9, q^{10}}\right)_\infty,
\]

\[
\sum_{n \geq 0} u_{2n+1} q^n = -\left(\frac{q^2, q^3, q^7, q^8}{q^4, q^5, q^5, q^6, q^{10}}\right)_\infty.
\]

For the 4-dissection, we have the eight conjectures of Hirschhorn, proved by Lewis and Liu,

\[
\sum_{n \geq 0} v_{4n} q^n = \left(\frac{q^2, q^7, q^3, q^7, q^8, q^8, q^8, q^8, q^{12}, q^{12}, q^{12}, q^{12}, q^{13}, q^{17}, q^{18}, q^{18}}{q^4, q^4, q^5, q^5, q^5, q^6, q^6, q^6, q^{14}, q^{14}, q^{15}, q^{15}, q^{15}, q^{16}, q^{16}, q^{20}}\right)_\infty.
\]