
INTRODUCTION

By the present time, one of the most popular classes of nonlinear wave processes has been soliton models of integrable and nonintegrable Hamilton systems. However, reduction of actual complex natural and technological systems to the Hamilton systems oversimplifies matters. Dissipation is an integral property of any system that evolves with time. The distributed dissipative systems are more complicated than the Hamilton systems, since in addition to the dispersion and nonlinearity they exchange energy with the surrounding medium. In this case, an additional balance between the energy inflow and outflow is required in order that the stationary soliton waves can exist. It has been elucidated that the stationary soliton waves are formed in liquid media with distributed dissipative gas bubbles.

The dissipation of the kinetic energy of radial bubble oscillations is caused by irreversibility or nonpolytropy of the processes that occur in the gas. It is assumed that during compression when the gas temperature is higher than the temperature of the surrounding liquid, the gas transferred a greater amount of heat compared to that it receives from the liquid during expansion when its temperature turns out to be lower than the temperature of the liquid [2]. According to this simplified concept, the thermal dissipation can be taken into account through the effective viscosity in the Rayleigh–Lamb equation [1] or through the effective dimensionless Nusselt number [2]. It was demonstrated that this approximation was inapplicable to acoustic problems, since they a priori suggested the thermodynamic irreversibility of heat exchange between the phases.

FORMULATION. MATHEMATICAL MODEL OF THE MEDIUM WITH BUBBLES

A study of acoustic and nonlinear wave propagation in gas-liquid systems with a correct consideration of heat dissipation by bubbles is of theoretical and practical interest. It can be described by a system of equations derived in [3] that considers the thermal conductivity of the gas phase:

\[
\frac{\gamma - 1}{\gamma} \frac{P_g}{T} \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} \right) - \frac{1}{r^2} \frac{\partial}{\partial r} \left( \kappa(T)r^2 \frac{\partial T}{\partial r} \right), \quad 0 < r < R, \quad t > 0, \tag{1}
\]
\[
\frac{dP_g}{dt} = \frac{3}{R} \left( (\gamma - 1) \kappa(T) \frac{\partial T}{\partial r} \bigg|_{r=R} - \gamma P_g \dot{R} \right),
\]

\[
\left(1 - \frac{\dot{R}}{c_L} \right) R \ddot{R} + \frac{3}{2} \dot{R}^2 \left(1 - \frac{\dot{R}}{3c_L} \right) = \frac{1}{c_L} \left( 1 + \frac{\dot{R}}{c_L} \right) \frac{1}{\rho} \left[ P_B(t) - P_S(t + \frac{R}{c_L}) - P_0 \right] + \frac{R}{c_L \rho} \frac{dP_B(t)}{dt},
\]

\[
\frac{\partial^2 \rho}{\partial t^2} - c_L^2 \frac{\partial^2 \rho}{\partial x^2} = \frac{\partial}{\partial t} \left( \rho \frac{\partial}{\partial t} \ln \left( 1 + \frac{\varphi_0}{1 - \varphi_0} \rho R^3 \right) \right),
\]

\[
P - P_0 = c_L^2 (\rho - \rho_0),
\]

\[
\varphi = \frac{4}{3} \pi n R^3.
\]

Here \( u = \frac{1}{\gamma P} \left( (\gamma - 1) \kappa(T) \frac{\partial T}{\partial r} \right) \), \( P_B = P_g \left( \frac{2\sigma}{R} - \frac{4\mu \dot{R}}{R} \right) \). \( t \) is time, \( r \) is the radial coordinate, \( T \) is the gas temperature, \( u \) specifies the radial velocity field in the gas, \( \kappa(T) \) is the thermal conductivity, \( \gamma = C_p / C_v \) is the ratio of heat capacities at constant pressure and constant volume, \( P_g \) is the kinetic gas pressure inside the bubble, \( P_B \) is the pressure on the external bubble surface, and \( P_S \) is the pressure created by an external sound perturbation. Other designations are standard [3]. Dots atop the variables mean the derivative with respect to time, bars atop them mean dimensionless quantities, and the subscript “0” indicates the corresponding parameters in the unperturbed state. From newly introduced assumptions, we note that the pressure inside the bubble is set homogeneous and depends only on time, since in our cases the velocity of bubble walls is much less than the sound velocity in the gas. The gas is considered ideal with a constant heat capacity. It is inert to the surrounding liquid, and the working pressure is much less than the gas condensation pressure. Under these assumptions, the mass exchange between the carrying and disperse phases can be neglected.

Equation of thermal conductivity (1) describes the law of non-stationary heat exchange between the spherical gas void and the liquid. Ordinary differential equation (2) for the gas pressure, derived under the above-described model assumptions with the above-indicated boundary conditions, takes into account heat dissipation. The condition of simultaneous deformation of phases relates the liquid and gas pressures and the bubble radius. It is described by the Rayleigh–Lamb equation or its modification (3) describing the monopole oscillations of a single spherical bubble in an infinite liquid. The wave propagation in the medium is described by generalized inhomogeneous Lighthill wave equation (4). This equation was corrected for the gas content caused by the compressibility of the carrying liquid. Formula (5) is the equation of state for the liquid in the acoustic approximation. Equation (6) relates the gas phase content \( \varphi \), bubble concentration \( n \), and bubble radius \( R \). The distinctive features of the examined model are the linearized hydrodynamic equations; the nonlinear and dispersion effects are engendered by the gas bubbles.

**Boundary and initial conditions**

For bubble oscillations, the problem of conjugate nonstationary heat exchange is solved with the corresponding boundary conditions specified at the moving liquid-gas interface. A large difference between the thermal physical parameters of the liquid and gas allows us to consider that the liquid is a thermostat, and its temperature is constant in the entire volume up to the phase interface. The boundary conditions for Eq. (1) are determined by a finite temperature in the center (the Neumann condition) and its equality to the liquid temperature on the wall (the Dirichlet condition):

\[
\frac{\partial T(0,t)}{\partial r} = 0, \quad T(R,t) = T_L.
\]