THE RELATIVISTIC INTERQUARK POTENTIAL

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A method for the calculation of interaction potentials in a momentum space is proposed which is based on exact calculations of the Lorentz structures of the corresponding interaction amplitudes. With this method the kernel of the QCD-motivated potential of a quark-antiquark relativistic system is obtained for different masses and an arbitrary angular momentum J. The calculation is performed taking into account the anomalous chromodynamic moments of the quarks. The case of the singlet state of a relativistic quark system is considered.

INTRODUCTION

The study of bound states in elementary particle physics is one of the most important and efficient techniques for the investigation of the properties of system constituent particles, including the dynamics of their interaction. This method is particularly attractive for describing mesons and baryons in the context of a composite quark model, since quarks are not observed in a free state.

There are many models intended to describe the properties of mesons. The general way of describing bound systems in relativistic quantum field theory is based on the Bethe–Salpeter four-dimensional covariant equation [1]. However, the difficulties encountered in using this method have led to a variety of approaches which can be considered as reductions of the Bethe–Salpeter equation. Various forms of these reductions can be found in the works by Salpeter [2], Logunov and Tavkhelidze [3], Kadyshesvky [4], Todorov [5], Gross [6], Keister and Polyzou [7], and others.

The major problem in models of this type is to determine the interaction potential of the quarks and to calculate the spectrum of masses of the bound system. Though the QCD-motivated potential proposed to describe quarkonia with heavy quarks [8] has been known for more than 30 years, the problem of finding a relativistic interquark potential is still not completely solved.

The goal of the present work was to obtain the kernel of the radial equation of a quark–antiquark relativistic system with an arbitrary angular momentum with the use of the amplitude of the single-gluon exchange and the demand of quark confinement in the context of the Poincare-covariant model based on relativistic Hamiltonian dynamics (RHD). In contrast to other works [9], we calculate exactly the kernel in the momentum space instead of doing expansions in the quark velocities. Thus, we consider all possible relativistic effects in this approximation. The knowledge of the kernel of the RHD integral equation makes it possible to switch to the procedure of numerical solution and calculation of the spectrum of masses of the quarkonium.

1. DESCRIPTION OF A QUARKONIUM IN TERMS OF RHD

In the RHD-based Poincare-covariant quark model, the "creation" of a relativistic bound system is begun with constructing a two-particle system of quarks with momenta \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) and masses \( m_q \) and \( m_Q \), which is described with the use of unitary representations of the Poincare group. Then an interaction \( \hat{V} \) is introduced so that the requirement of Poincare invariance be satisfied for the system of interacting particles. This requirement is realized in the form of a Poincare algebra on the manifold of observable dynamic systems.

To separate the relative motion and the motion of the center of inertia in the Poincare-covariant model, the quark momenta \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) can be transformed to a total momentum \( \mathbf{P} \) and a relative momentum \( \mathbf{k} \):
\[ P = p_1 + p_2, \quad k = \frac{1}{2} (p_1 - p_2) + \frac{P}{M_0} \left( \frac{m_q^2 - m_{\overline{q}}^2 - M_0 \left[ \omega_{m_q} (p_2) - \omega_{m_q} (p_1) \right]}{\omega_{m_q} (P) + M_0} \right). \] (1)

In equation (1) we have introduced an invariant mass of two noninteracting quarks:

\[ M_0 = \omega_{m_q} (k) + \omega_{m_{\overline{q}}} (k), (k = |k|), \] (2)

with the function \( \omega_m (p) = \sqrt{m^2 + p^2} \).

The operator of mass \( M \) for a bound system with momentum \( Q \), spin \( J \), and projected spin \( \mu \) is the sum of the operator \( M_0 \) (2) and phenomenological confining potential \( V \). In the general case, the wave function of a bound system of spinor quarks in RHD satisfies the three-dimensional integral equation [7]

\[ \sum_{\lambda_1, \lambda_2} \int < k, \sigma_1, \sigma_2 || \hat{V} || k', \lambda_1, \lambda_2 > \Psi_{Q, \lambda_1}^{J, \mu} (k') dk' = (M_{\text{meson}} - M_0) \Psi_{Q, \sigma_1, \sigma_2} (k) \] (3)

with a reduced matrix element and eigenvalues \( M_{\text{meson}} \).

For deriving the one-dimensional radial equation we use the Clebsch–Gordan expansion of the Poincare group. Since it is convenient to use the spiral states of quarks for analytic calculation of the kernel, and the bound states are classified using the orbit moment \( L \), total spin \( S \), and total angular momentum \( J \), we shall perform the calculation of the kernel in two stages. At the first stage, we construct, in the system of the center of inertia \( Q = 0 \), a state with quantum numbers \( J \) and \( \mu \) and with quark helicities \( \lambda_1 \) and \( \lambda_2 \), which form the basis of a nonreducible two-particle space of the Poincare group \( |Q = 0, J, \mu, k, \lambda_1, \lambda_2 \rangle \). At the second stage, with the help of the Clebsch–Gordan coefficients of the group of rotations, \( C_{L_1 S J}^{L_2 1/2 1/2} \) and \( C_{0 \lambda \lambda}^{LS \lambda \lambda} \), we obtain a basis of states with quantum numbers \( J, \mu, L, \) and \( S \):

\[ |k, J, \mu, L, S \rangle = \sum_{\lambda_1, \lambda_2} \frac{2L + 1}{2J + 1} C_{L_1 \lambda_1 - \lambda_2 \lambda}^{LS \lambda \lambda} C_{0 \lambda \lambda}^{LS \lambda \lambda} |k, J, \mu, L, S \rangle. \] (4)

The radial equation for a two-particle bound state in the system of the center of inertia has the form

\[ \sum_{L, S} \int_{L, S} V_{L, S}^{J, \mu} \langle k', k || \hat{V} || k, J, \mu, L, S > \Psi_{L, S}^{J, \mu} (k') k'^2 dk' = (M_{\text{meson}} - M_0) \Psi_{L, S}^{J, \mu} (k), \] (5)

where the operator \( V_{L, S}^{J, \mu} \) is determined by the relation

\[ V_{L, S}^{J, \mu} (k', k) = \int \left( \cos \theta \right) \int d\phi \left( \cos \beta \right) \cos \phi \beta (\phi, \beta, -\phi) \langle k', \lambda_1, \lambda_2 \| \hat{V} \| k, \lambda_1, \lambda_2 \rangle \] (6)

and the matrix element \( V_{L_1 S_1 \lambda_1 \lambda_2}^{J, \mu} (k', k) \) is found by evaluating the integral

\[ V_{L_1 S_1 \lambda_1 \lambda_2}^{J, \mu} (k', k) = \frac{1}{2\pi} \int d\phi \int d\beta \frac{2\pi}{0} \left( \cos \beta \right) \left( \cos \theta \right) \cos \phi \beta (\phi, \beta, -\phi) \langle k', \lambda_1, \lambda_2 \| \hat{V} \| k, \lambda_1, \lambda_2 \rangle, \] (7)