THE VOLterra PROPERTY OF SOME PROBLEMS WITH THE BITSADZE–SAMARSKIĬ-TYPE CONDITIONS FOR A MIXED PARABOLIC-HYPERBOLIC EQUATION

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Abstract: We consider problems with the Bitsadze–Samarskiĭ-type conditions for a mixed parabolic-hyperbolic equation with noncharacteristic type change curve. We prove theorems on the unique existence of regular and strong solutions and the Volterra property for the problems under consideration.

Keywords: parabolic-hyperbolic equation, Bitsadze–Samarskiĭ problem, solvability, Volterra property

In 1969 A. V. Bitsadze and A. A. Samarskiĭ [1] stated and studied a new problem for a uniformly elliptic equation. The distinction of this problem from the others is that the boundary values of a solution are repeated at the interior points of the domain where the sought function satisfies the equation. Their article gave impetus to a series of articles on various statements of the Bitsadze–Samarskiĭ-type problems for many equations. Among them we mention the articles by V. A. Il’in and E. I. Moiseev [2], M. S. Salakhitdinov and A. K. Urinov [3], and many others.

In spite of a rich variety of the articles on the Bitsadze–Samarskiĭ-type problems, in the mathematical literature we find no problems for a mixed-type equation with a condition connecting the values of a solution on a characteristic and an arbitrary monotone curve lying strictly inside the characteristic triangle.

The following natural questions arise in this connection: Can we state similar problems for mixed parabolic-hyperbolic equations with noncharacteristic type change curve? Are there any Volterra problems among these problems?

The spectral properties (in particular, the Volterra property) of the boundary-value problems for a parabolic-hyperbolic equation were studied in [4–8].

This article is devoted to studying one class of problems with the Bitsadze–Samarskiĭ-type conditions for a second-order mixed parabolic-hyperbolic equation with two independent variables.

The main result of the article is the proof of the Volterra property for the problems with the Bitsadze–Samarskiĭ-type conditions for a mixed parabolic-hyperbolic equation with noncharacteristic type change curve.

Consider the equation

\[ Lu = f(x, y), \]  

where

\[ Lu = \begin{cases} u_x - u_{yy}, & y > 0, \\ u_{xx} - u_{yy}, & y < 0, \end{cases} \]  

in a bounded simply connected domain \( \Omega \) of the \( x, y \)-plane which is bounded by some segments \( AA_0, A_0B_0, \) and \( BB_0 \) of the straight lines \( x = 0, \) \( y = 1, \) and \( x = 1 \) for \( y > 0 \) and by the characteristics \( AC: x + y = 0 \) and \( BC: x - y = 1 \) of (1) for \( y < 0. \)

Suppose that a smooth curve \( AD: y = -\gamma(x), \) \( 0 < x < l, \) where \( 0.5 < l < 1, \) \( \gamma(0) = 0, \) and \( l + \gamma(l) = 1, \) lies inside the characteristic triangle \( 0 \leq x + y \leq x - y \leq 1. \) Considering the curve \( AD, \) we henceforth suppose that \( \gamma(x) \) is twice continuously differentiable while \( x - \gamma(x) \) and \( x + \gamma(x) \) are monotone increasing; moreover, \( 0 < \gamma'(0) < 1 \) and \( \gamma(x) > 0, x > 0. \)

Put \( \Omega_0 = \Omega \cap \{y > 0\} \) and \( \Omega_1 = \Omega \cap \{y < 0\}, \) and denote by \( W_2^1(\Omega) \) the Sobolev space with inner product \( (\cdot, \cdot)_I \) and norm \( \| \cdot \|_I \) and by \( W_2^0(\Omega) \equiv L_2(\Omega), \) the space of square summable functions on \( \Omega. \)
Problem $TM_1$. Find a solution to (1) satisfying the conditions

$$u|_{AA_0}\cup AB_0 = 0,$$

$$[u_x - u_y](\theta(t)) + \mu(t)[u_x - u_y](\theta^*(t)) = 0,$$

where $\theta(t) (\theta^*(t))$ is the affix of the intersection point of the characteristic $AC$ (the curve $AD$) and the characteristic starting at the point $(t, 0)$, $0 < t < 1$, and $\mu(t)$ is a given function.

In the case of $\mu(t) \equiv 0$ Problem $TM_1$ coincides with the Tricomi problem for a parabolic-hyperbolic equation with a noncharacteristic type change curve. In this case Problem $TM_1$ was considered in [9] (regular solvability), [10] (strong solvability), and [8] (uniqueness of a solution to the problem for an equation with complex coefficients); when $\mu(t) = \infty$, i.e., the conditions $u_x - u_y|_{AD} = 0$ are given on the curve $AD$, strong solvability and the Volterra property of Problem $TM_1$ were studied in [7].

Problem $TM_2$. Find a solution to (1) satisfying (3) and

$$[u_x - u_y](\theta(t)) + \mu(t)[u_x + u_y](\theta^*(t)) = 0.$$  \hspace{1cm} (4.2)

Problem $TM_3$. Find a solution to (1) satisfying (3) and

$$\frac{d}{dt}u[\theta(t)] + \mu(t)\frac{d}{dt}u[\theta^*(t)] = 0.$$  \hspace{1cm} (4.3)

Observe that if $\mu(t) = \mu = \text{const}$ then (4.3) are equivalent to the condition

$$u[\theta(t)] + \mu(t)u[\theta^*(t)] = 0$$

which gives a pointwise connection between the values of a solution on the characteristic and the values of a solution on some curve strictly inside the domain.

By a regular solution to Problem $TM_i$ ($i = 1, 2, 3$) we mean a function $u(x, y) \in W$, with

$$W = C(\overline{\Omega}) \cap C^1(\Omega \cup AC) \cap C^{1,2}(\Omega_0) \cap C^{2,2}(\Omega_1),$$

which satisfies (1) in $\Omega_0 \cup \Omega_1$ and (3) and (4i) ($i = 1, 2, 3$).

A function $u(x, y) \in L_2(\Omega)$ is called a strong solution to Problem $TM_i$ ($i = 1, 2, 3$) if there is a sequence $\{u_n\}$ of $u_n \in W$ satisfying (3) and (4i) ($i = 1, 2, 3$) and such that $\|u_n - u\|_0 \rightarrow 0$ and $\|Lu_n - f\|_0 \rightarrow 0$ as $n \rightarrow \infty$.

**Theorem 1.1.** Let $\mu(t) \in C^2[0, 1]$ and $\mu(t) \neq -1$, $0 \leq t \leq 1$. Then, for every function $f(x, y) \in C^1(\overline{\Omega})$, with $f(A) = 0$, Problem $TM_1$ has a unique regular solution satisfying the inequality

$$\|u\|_1 \leq c\|f\|_0$$  \hspace{1cm} (5)

and representable as

$$u(x, y) = \iint_\Omega K_i(x, y; x_1, y_1)f(x_1, y_1)dx_1dy_1, \quad i = 1, 2, 3,$$  \hspace{1cm} (6i)

where $K_i(x, y; x_1, y_1) \in L_2(\Omega \times \Omega)$ (for Problem $TM_1$ $i = 1$).

In (5) and below we denote by $c$ a positive constant independent of $u(x, y)$ and not necessarily the same in various formulas.

**Theorem 1.2.** Suppose that the conditions of Theorem 1.1 are satisfied. Then, for each function $f(x, y) \in L_2(\Omega)$, Problem $TM_1$ has a unique strong solution. This solution belongs to the class

$$C(\overline{\Omega}) \cap W_2^1(\Omega) \cap W_2^{1,2}(\Omega_0),$$  \hspace{1cm} (7)

satisfies (5), and is representable in the form (6i).

Unlike Problem $TM_1$, the Volterra property and well-posedness of Problem $TM_2$ depend crucially on the relation between the “contraction” $\mu(0)$ at the origin of the derivative in the direction of the characteristic $BC$ and the polar angle $\alpha$ of $AD$ and the abscissa axis.