ON SOME PROPERTIES OF FUNCTIONS IN THE SOBOLEV–MORREY-TYPE SPACES $W^l_{p,a,\kappa,\tau}(G)$

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Abstract: In the framework of embedding theory, we study some properties of the functions of the Sobolev–Morrey-type spaces and local smoothness of solutions to one class of quasielliptic equations.

Keywords: Sobolev–Morrey-type space, flexible \( \lambda \)-horn, embedding theorem, integral representations, Hölder condition, smoothness

This article is devoted to studying the problems of embedding for the Sobolev–Morrey-type spaces $W^l_{p,a,\kappa,\tau}(G)$ \( l \in \mathbb{N}^n, p \in [1,\infty)^n, a \in [0,1]^n, \kappa \in (0,\infty)^n \), and \( \tau \in [1,\infty] \). Using integral representations for functions on a domain satisfying the flexible \( \lambda \)-horn condition which was introduced by O. V. Besov (see [1]), we prove some embedding theorems for the spaces under consideration.

In the case \( \tau = \infty, a = (a,\ldots,a) \), and \( p = (p,\ldots,p) \) the Sobolev–Morrey spaces $W^l_{p,a,\kappa,\infty}(G)$ were defined and studied by V. P. Il’in [2] and in the case \( 1 \leq \tau < \infty, a = (a,\ldots,a) \), and \( p = (p,\ldots,p) \) the Sobolev–Morrey-type spaces $W^l_{p,a,\kappa,\tau}(G)$ were introduced by V. S. Guliev.

Also, we apply the obtained embedding theorems to studying the local smoothness of solutions to some classes of quasielliptic equations. The Hölder continuity of solutions to quasielliptic equations with continuous or Hölder continuous coefficients of the leading derivatives was considered in [3]. In [4] $L_p$ estimates for solutions were studied, under the condition that the coefficients of the leading derivatives are infinitely differentiable, and in [5, 6] some other problems of the theory of quasielliptic equations were considered. In [7] a theorem was proven claiming that the solution belongs to the Hölder class inside the domain, and in [8] local “interior” Hölder estimates were obtained for solutions to a quasielliptic-type equation in the case when the right-hand side satisfies the anisotropic Hölder condition. In this article, as in [7], we study the Hölder continuity of a solution without any smoothness conditions on \( a_\alpha \beta (x) \).

However, observe that unlike [7] here

1. the Hölder “exponent” is greater than that in [7];
2. \( f_\alpha \) for \( (\alpha, \lambda) = 1 \) belongs to a broader class; i.e., \( f_\alpha \in L_{2, a, \kappa}(G) \).

Let \( G \) be a domain of \( \mathbb{R}^n \), \( t > 0 \). Given \( x \in \mathbb{R}^n \), we put

\[
I_{t \tau}(x) = \{ y : |y_j - x_j| < (1/2)t^{\kappa_j}, j = 1,2,\ldots,n \}, \quad G_{t \tau}(x) = G \cap I_{t \tau}(x).
\]

Denote by $W^l_{p,a,\kappa,\tau}(G)$ the space of locally summable functions \( f \) on \( G \) having the weak derivatives $D^l_i f$ on \( G \) \( (i = 1,2,\ldots,n) \) with the finite norm

\[
\|f\|_{W^l_{p,a,\kappa,\tau}(G)} = \|f\|_{L_{p,a,\kappa,\tau}(G)} + \sum_{i=1}^n \|D^l_i f\|_{L_{p,a,\kappa,\tau}(G)}, \quad 1 \leq \tau < \infty, \quad (1)
\]

\[
\|f\|_{W^l_{p,a,\kappa,\infty}(G)} = \|f\|_{L_{p,a,\kappa,\infty}(G)} + \sum_{i=1}^n \|D^l_i f\|_{L_{p,a,\kappa,\infty}(G)},
\]

where

\[
\|f\|_{L_{p,a,\kappa,\tau}(G)} = \|f\|_{L_{p,a,\kappa,\tau,G}} = \sup_{x \in G} \left\{ \int_0^{t_0} \left[ \sum_{j=1}^n \frac{\kappa_j}{r_j} \right] \frac{\|f\|_{L_{p,a,\kappa,\tau}(G)}}{t} dt \right\}^{1/\tau}, \quad 1 \leq \tau < \infty, \quad (2)
\]
\[ \|f\|_{L_p; a, \infty} = \|f\|_{L_p, \infty; G} = \|f\|_{L_p, \infty; G} = \sup_{x \in G, t > 0} \left( [t]_1 \left[ \frac{-\sum_{j=1}^{n} \kappa_j a_j}{\eta} \int_{G_{\kappa}(x)} |f(y)|^p \, dy \right]^\frac{p}{\eta} \right), \]

\[ \|f\|_{p, a, \infty} = \left\{ \begin{array}{l}
\int_{G_{\kappa}(x_1)} \left( \cdots \int_{G_{\kappa}(x_2)} \left( \int_{G_{\kappa}(x_3)} |f(y)|^p \, dy \right)^\frac{p_1}{p_2} \, dy_2 \right)^\frac{p_2}{p_3} \cdots \right)^\frac{p_{n-1}}{p_n}, \end{array} \right. \]

\([t]_1 = \min \{1, t\}, \) and \(t_0\) is a fixed positive number.

Henceforth we write \(p \leq q\) (\(p < q\)), where \(p = (p_1, \ldots, p_n)\) and \(q = (q_1, \ldots, q_n)\), if \(p_j \leq q_j\) and \(p_j < q_j\) \((j = 1, \ldots, n)\); in particular, \(1 \leq p \leq \infty\) \((1 = (1, \ldots, 1)\) and \(\infty = (\infty, \ldots, \infty)\) means that \(1 \leq p_j \leq \infty\) \((j = 1, \ldots, n)\).

Observe some properties of \(L_{p, a, \kappa}(G)\) and \(W^l_{p, a, \kappa}(G)\).

1. The norms of the form (2) are equivalent for various \(t_0\), \(0 < t_0 < \infty\).

2. The following embeddings hold for arbitrary \(\kappa > 0\) and \(0 \leq a \leq 1\):

\[ L_{p, a, \kappa}(G) \hookrightarrow L_{p, a}(G), \quad W^l_{p, a, \kappa}(G) \hookrightarrow W^l_{p, a}(G); \]

i.e.,

\[ \|f\|_{p, a, \kappa} \leq C \|f\|_{p, a, \kappa}(G) \tag{3} \]

and

\[ \|f\|_{W^l_{p, a, \kappa}} \leq C \|f\|_{W^l_{p, a, \kappa}(G)}. \tag{4} \]

We first prove (3). Let \(p_1 = p_2 = \cdots = p_n = p\). Then

\[ \|f\|_{p, a, \kappa} = \sup_{x \in G} \int_0^{[t]_1} \left( \int_{G_{\kappa}(x)} |f(y)|^p \, dy \right)^\frac{p}{\eta} \, d\eta \]

\[ \geq \sup_{x \in G, 0 < t < t_0} \int_0^{[t]_1} \left( \int_{G_{\kappa}(x)} |f(y)|^p \, dy \right)^\frac{p}{\eta} \, d\eta \]

\[ \geq \sup_{x \in G, 0 < t < t_0} \left( \int_{G_{\kappa}(x)} |f(y)|^p \, dy \right)^\frac{p}{\eta} \int_0^{[t]_1} \, d\eta \]

\[ \geq C \sup_{x \in G, 0 < t < t_0} \left( \int_{G_{\kappa}(x)} |f(y)|^p \, dy \right)^\frac{p}{\eta} = C \|f\|_{p, a, \kappa}. \tag{5} \]

The inequality for the vectors \(p = (p_1, p_2, \ldots, p_n)\) is obtained by successive application of (5) to each single variable. Similarly, we prove (4).

3. The spaces \(L_{p, a, \kappa}(G)\) and \(W^l_{p, a, \kappa}(G)\) are complete.

4. For every real \(c > 0\),

\[ \|f\|_{p, a, \kappa; G} = c^{-\frac{1}{p}} \|f\|_{p, a, \kappa; G}, \quad \|f\|_{W^l_{p, a, \kappa}(G)} = c^{-\frac{1}{p}} \|f\|_{W^l_{p, a, \kappa}(G)}. \]

5. The following relations are valid for every \(\kappa > 0\):

(a) \(\|f\|_{p, 0, \kappa; G} = \|f\|_{p; G}, \quad \|f\|_{W^l_{p, 0, \kappa}(G)} = \|f\|_{W^l_{p; G}}\);

(b) \(\|f\|_{p, 1, \kappa; G} \geq \|f\|_{\kappa; G}, \quad \|f\|_{W^l_{p, 1, \kappa}(G)} \geq \|f\|_{\kappa}(G)\).