Consider two numberings \( \nu \) and \( \mu \) of some nonempty at most countable set \( S \). Say that \( \nu \) is \( e \)-reducible to \( \mu \) if there exists a mapping \( \Phi \) such that

\[(\forall s \in S)(\nu^{-1}(s) = \Phi(\mu^{-1}(s))).\]

A mapping \( \Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \) is called an \( e \)-operator if there exists a recursively enumerable set (RES) \( W \) such that

\[(\forall X \subseteq \mathbb{N})(\Phi(X) = \{x : (\exists y)(\langle x, y \rangle \in W \land D_y \subseteq X)\}),\]

where \( \langle x, y \rangle \) is the Cantor index of the pair \( (x, y) \) of nonnegative integers, \( D_y \) is the finite subset of the set \( \mathbb{N} \) of nonnegative integers with the canonical index \( y \), and \( \mathcal{P}(\mathbb{N}) \) is the set of all subsets of \( \mathbb{N} \).

A mapping \( \Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \) is called a \( p \)-operator if there exists a general recursive function (GRF) \( f \) such that

\[(\forall X \subseteq \mathbb{N})(\Phi(X) = \{x : (\exists y)(y \in D_{f(x)} \land D_y \subseteq X)\}).\]

Below in the case of \( e \)-reducibility of numberings we will identify the \( e \)-operator \( \Phi \) with the RES \( W \), meaning that \( \nu \) is \( e \)-reducible to \( \mu \) by \( W \). Similarly in the case of \( p \)-reducibility of numberings we will identify the \( p \)-operator \( \Phi \) with the GRF \( f \), meaning that \( \nu \) is \( e \)-reducible to \( \mu \) by \( f \).

The classes of \( e \)-equivalent (computable) numberings of some set \( S \) form \( e \)-degrees. Equipped with the partial order induced by the relation \( \leq_e \), they form an upper semilattice \( \mathcal{L}_e(S) \) of (computable) numberings with the top element \( \mu(x) \oplus \nu(x) \). Similarly we define the \( p \)-degrees of numberings of \( S \) and the upper semilattice \( \mathcal{L}_p(S) \). It is known that \( p \)-reducibility is “stronger” than \( e \)-reducibility. Thus, an \( e \)-degree consists in general of several \( p \)-degrees. The corresponding results are presented in the article [1], describing the semilattices \( \mathcal{L}_e(\mathcal{F}) \) for finite families \( \mathcal{F} \) of RES’s and exhibiting an example of a computable family of RES’s with no \( e \)-principal numbering.

A family \( \mathcal{R} \) of RES’s is called discrete if there exists a family \( \mathcal{F} \) of finite RES’s such that

1. given \( D \in \mathcal{F} \) there exists at most one \( R \in \mathcal{R} \) with \( D \subseteq R \);
2. given \( R \in \mathcal{R} \) there exists \( D \in \mathcal{F} \) with \( D \subseteq R \).

A discrete family \( \mathcal{R} \) of RES’s is called effectively discrete if for some GRF \( f \) there exists a strongly enumerable family \( \mathcal{F} \) of finite sets such that \( \mathcal{F} = \{D_{f(0)}, D_{f(1)}, \ldots \} \).

It is proved in [2] that all computable numberings of an effectively discrete family \( \mathcal{R} \) of RES’s are \( m \)-equivalent and form the trivial semilattice \( \mathcal{L}_e(\mathcal{R}) \).

**Proposition 1.** There exists a discrete but not effectively discrete family \( \mathcal{R} \) of finite sets such that the semilattice \( \mathcal{L}_e(\mathcal{R}) \) is a singleton.

**Proof.** Take some nonrecursive RES \( R \) and consider the family \( \mathcal{R} = \{R_0, R_1, \ldots, R_n, \ldots\} \) of pairwise disjoint sets

\[R_0 = [0, a_0], R_1 = (a_0, a_1], \ldots, R_n = (a_{n-1}, a_n], \ldots,\]
where \( \{a_n\}_{n \geq 0} \) is the direct numbering of the complement to \( R \). The family \( \mathcal{R} \) is computable.

Indeed, the numbering
\[
\nu(x) = \begin{cases} [0, a_0] & \text{for } x \leq a_0, \\ (a_n, a_{n+1}] & \text{for } x \in (a_n, a_{n+1}] \end{cases}
\]
is computable. In order to verify this, enumerate the nonrecursive RES \( R \) step-by-step. At step \( t = 1 \) put \( \nu^1(x) = \{x\} \). If at step \( t \geq 1 \) some elements \( a \) and \( b \) turn out to be the minimal and maximal elements of the numbering \( \nu^t(x) \), and the elements \( a - 1 \) or \( b \) in \( R \) have been computed before this step, then put 
\[
\nu^{t+1}(x) = \nu^t(x) \cup \{a - 1\} \text{ or } \nu^{t+1}(x) = \nu^t(x) \cup \{b + 1\}.
\]

Let us now address the question whether \( \mathcal{R} \) is effectively discrete.

Given \( \mathcal{R} \), suppose that there exists a strongly computable family \( \mathcal{F} \) of finite sets confirming the effective discreteness of \( \mathcal{R} \). Since \( \mathcal{R} \) consists of pairwise disjoint sets, we may assume that all sets in \( \mathcal{F} \) are singletons. This means that there exists a computable sequence \( a_0, a_1, \ldots \) such that for every \( x \in R \) there exists a unique index \( n \in \mathbb{N} \) with \( a_n \in \nu(x) \). Enumerate the elements of \( \nu(x) \) until for \( a_n \) we find \( a_k, a_l, b_k, b_l, y, \) and \( z \) such that \( a_k \leq a_l \) and \( b_k \leq b_l \). Suppose computed in \( R \) at some step \( t \) the elements \( a_k \in \nu^t(y), b_k \in \nu^t(y), a_k \in \nu^t(z), b_k \in \nu^t(z), a_n \in (b_k, b_l), \) and also all elements of \( (b_k, b_l) \) but \( w \). It is clear that \( w \in \mathbb{N} \setminus R \).

Thus, the complement to the nonrecursive RES \( R \) turns out to be a RES, which by Post’s theorem contradicts the nonrecursiveness of the RES \( R \).

Finally, take another computable numbering \( \mu \) of \( \mathcal{R} \). For all \( x \geq 0 \) enumerate the elements of the numberings \( \nu(x) \) and \( \mu(x) \) until we find some \( y \) with \( \mu(x) \cap \nu(y) \neq \emptyset \). It is obvious that \( \mu(x) = \nu(y) \) and the GRF \( f(x) = y \) reduces \( \mu \) to \( \nu \). In this case, every numbering \( \mu \) of \( \mathcal{R} \) is positive because all sets in \( \mathcal{R} \) are pairwise disjoint. More exactly, the relation \( x y \Leftrightarrow \mu(x) = \nu(y) \) is a positive equivalence. □

Consider some family \( \mathcal{F} \) of GRFs. Call a limit point for \( \mathcal{F} \) some GRF \( f \) such that for every \( n \in \mathbb{N} \) the family \( \mathcal{F} \) contains some GRF \( g \) with \( (\forall x \leq n)(f(x) = g(x)) \). If \( \mathcal{F} \) contains no limit points then it is discrete.

**Proposition 2.** Given some nonrecursive RES \( R \) we can define a discrete family \( \mathcal{F} \) of GRFs such that \( \mathcal{L}_e(\mathcal{F}) \) contains the top element and countably many minimal elements.

**Proof.** Take some GRF \( f \) that enumerates \( R \) without repetitions. For all \( n \in \mathbb{N} \) put
\[
\nu_{2n}(x) = n,
\]
\[
\nu_{2n+1}(x) = \begin{cases} n & \text{if } x = 0 \text{ or } n \notin \{f(y) : y \leq x\}, \\ 0 & \text{otherwise.} \end{cases}
\]

Here \( \nu_n(x) \) is the value of \( f \) with index \( n \) computed at \( x \). It is clear that the numbering \( \nu \) is a computable numbering of some discrete family \( \mathcal{F} \) of GRFs that is not positive. Otherwise \( R \) would be recursive because \( \nu \in \mathcal{R} \Leftrightarrow \nu_{2n} = \nu_{2n+1} \).

Given another numbering \( \mu \) of \( \mathcal{F} \) it follows that \( \mu \leq_e \nu \) by the \( e \)-operator \( \Phi \) consisting of the indices
\[
(1) \ (x, y) \text{ such that } D_y = \{2\mu_x(0)\};
\]
\[
(2) \ (x, z) \text{ such that } D_z = \{2\mu_x(0) + 1\}.
\]

Naturally, this is so under the assumption that there exists some index \( m \) with \( \mu_x(m) = 0 \). Thus, the numbering \( \nu \) of \( \mathcal{F} \) will be an \( e \)-principal numbering, and the \( e \)-degree of \( \nu \) is the top element of \( \mathcal{L}_e(\mathcal{F}) \).

On the other hand, the semilattice \( \mathcal{L}_m(\mathcal{F}) \) of all \( m \)-equivalent degrees of computable numberings of \( \mathcal{F} \) is nontrivial because \( \mathcal{F} \) always admits a single-valued computable numbering. Thus, the family \( \mathcal{F} \) admits countably many pairwise \( m \)-incomparable single-valued numberings [3], which are pairwise \( m \)-incomparable because \( \mu \leq_e \nu, \mu =_m \nu, \) and \( \nu \) is a positive numbering. □

Recall that \( \mathcal{L}_m(\mathcal{F}) \) is a nontrivial semilattice with no \( m \)-principal numbering [3]. There exist computable numberings of recursively enumerable sets that are \( e \)-equivalent to noncomputable numberings. Thus, of primary interest are the examples of computable numberings that are not positive and to which some computable numberings \( e \)-reduce.