FINITE GROUPS WITH $\mathcal{F}$-SUBNORMAL CONDITIONS

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Abstract: Let $\mathcal{F}$ be a subgroup-closed saturated formation. A finite group $G$ is called an $\mathcal{F}_{pc}$-group provided that each subgroup $X$ of $G$ is $\mathcal{F}$-subabnormal in the $\mathcal{F}$-subnormal closure of $X$ in $G$. Let $\mathcal{F}_{pc}$ be the class of all $\mathcal{F}_{pc}$-groups. We study some properties of $\mathcal{F}_{pc}$-groups and describe the structure of $\mathcal{F}_{pc}$-groups when $\mathcal{F}$ is the class of all soluble $\pi$-closed groups, where $\pi$ is a given nonempty set of prime numbers.

Keywords: $\mathcal{F}$-subnormal subgroup, $\mathcal{F}$-projector, $\mathcal{F}$-covering subgroup, $\mathcal{F}_{pc}$-group

1. Introduction. Let $\mathcal{F}$ be a saturated formation and let $G$ be a finite group. We let $\mathcal{N}$ denote the formation of nilpotent groups. In this article we study some interesting properties of $\mathcal{F}_{pc}$-groups. We recall the following definitions:

Definition 1 [1]. A maximal subgroup $M$ of $G$ is $\mathcal{F}$-normal in $G$ provided that $G/\text{core}_G(M) \in \mathcal{F}$ where $\text{core}_G(M)$ is the core of $M$ in $G$; otherwise $M$ is called $\mathcal{F}$-abnormal in $G$.

Definition 2 [1]. A subgroup $X$ of $G$ is called $\mathcal{F}$-subnormal in $G$ if, either $X = G$ or there exists a maximal chain:

$$X = U_0 < U_1 < \cdots < U_l = G$$

such that $U_{i-1}$ is $\mathcal{F}$-normal in $U_i$ for all $i = 1, 2, \ldots, l$; and $X$ is said to be $\mathcal{F}$-subabnormal in $G$ if $H$ is $\mathcal{F}$-abnormal in $K$ whenever $X \leq H < \cdot K \leq G$ where $H$ is maximal in $K$.

By definition, $G$ is both $\mathcal{F}$-subnormal and $\mathcal{F}$-subabnormal in $G$.

Definition 3 [1]. Let $F$ be an $\mathcal{F}$-subgroup of $G$.

1. $F$ is called an $\mathcal{F}$-projector if $FH/H$ is a maximal $\mathcal{F}$-subgroup of $G/H$ for all normal subgroups $H$ of $G$.

2. $F$ is called an $\mathcal{F}$-covering subgroup if $F \leq H$ implies $H = H^\mathcal{F}F$.

We denote by $\text{Proj}_\mathcal{F}(G)$ the set of all $\mathcal{F}$-projectors of $G$ and by $\text{Cov}_\mathcal{F}(G)$, the set of all $\mathcal{F}$-covering subgroups of $G$.

3. The intersection of all normal subgroups $N$ of $G$ satisfying $G/N \in \mathcal{F}$ is called the $\mathcal{F}$-residual of $G$ and denoted by $G^\mathcal{F}$.

Definition 4. Let $\Sigma(G)$ denote the set of all minimal supplements $L$ to $G^\mathcal{F}$ in $G$, that is, $G = G^\mathcal{F}L$ but $G > G^\mathcal{F}B$ for every proper subgroup $B$ of $L$.

In [2], Förster defined $\mathcal{F}_{an}$-groups to be the finite groups in which every subgroup is either $\mathcal{F}$-subnormal or $\mathcal{F}$-subabnormal. The groups in $\mathcal{F}_{an}$ were studied in [2–4] for some special $\mathcal{F}$. Shirong Li in [5] generalized the class of $\mathcal{F}_{an}$-groups by defining $\mathcal{F}_{pc}$-groups.

Definition 5 [5]. For a subgroup $X$ of $G$, $S_G(X)$ denotes the $\mathcal{F}$-subnormal closure of $X$ in $G$, the intersection of $\mathcal{F}$-subnormal subgroups of $G$ containing $X$.

Definition 6 [5]. A group $G$ is called an $\mathcal{F}_{pc}$-group if every subgroup $X$ of $G$ is $\mathcal{F}$-subabnormal in $S_G(X)$. We denote by $\mathcal{F}_{pc}$ the class of all $\mathcal{F}_{pc}$-groups.

Remarks. (1) $\mathcal{F}_{an} \subseteq \mathcal{F}_{pc}$.

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(2) For two saturated formations $\mathcal{F}$ and $\mathcal{H}$ satisfying $\mathcal{F} \subseteq \mathcal{H}$, in general, we have $\mathcal{F}_{pc} \subseteq \mathcal{H}_{pc}$.

(3) The concept of $\mathcal{F}_{pc}$-groups provides many interesting classes of groups. For example, the class $\mathcal{N}_{pc}$ consists of the groups $G$ whose every subgroup $X$ is abnormal in the subnormal closure of $X$ in $G$ (see Section 3). On the other hand, in [6] the following concept was introduced: A subgroup $X$ of $G$ is said to be an NE-subgroup if $N_G(X) \cap X^G = X$ where $X^G$ is the normal closure of $X$ in $G$. The groups whose subgroups are NE-subgroups belong to $\mathcal{N}_{pc}$ and it is showed by Yangming Li in [7] that these groups coincide with soluble T-groups (the groups in which the normality is transitive). Therefore, $\mathcal{N}_{pc}$-groups is a generalization of soluble T-groups. Another example is the case when $\mathcal{F}$ is the class of $p$-nilpotent groups, which was investigated in [5].

2. The main results.

**Lemma 1** [2]. Let $\mathcal{F}$ be a subgroup-closed saturated formation. Then

(1) If $H$ is an $\mathcal{F}$-subnormal subgroup of $G$ and $H \triangleleft K \leq G$ then $H$ is also $\mathcal{F}$-subnormal in $K$.

(2) If $H$ is $\mathcal{F}$-subnormal in $G$ and $N \triangleleft G$ then $HN/N$ is $\mathcal{F}$-subnormal in $H/G$.

**Lemma 2** [8]. Let $\mathcal{F}$ be a saturated formation and $F \in \Sigma(G)$. Then

(1) $F$ is an $\mathcal{F}$-subgroup of $G$.

(2) $F \in \text{Cov}_{\mathcal{F}}(G)$ if and only if $F$ is $\mathcal{F}$-subabnormal in $G$.

**Lemma 3.** Let $\mathcal{F}$ be a saturated formation. If $\Sigma(G) \subseteq \text{Proj}_{\mathcal{F}}(G)$ then $\text{Proj}_{\mathcal{F}}(G) = \Sigma(G) = \text{Cov}_{\mathcal{F}}(G)$.

**Proof.** We know that every $F \in \text{Proj}_{\mathcal{F}}(G)$ satisfies $G = FG^\mathcal{F}$, and so $F$ contains a minimal supplement to $G^\mathcal{F}$ in $G$, say $F_1$. By hypothesis, $F_1 \in \text{Proj}_{\mathcal{F}}(G)$. In particular, $F = F_1$, and hence $\text{Proj}_{\mathcal{F}}(G) = \Sigma(G)$.

Since $\text{Cov}_{\mathcal{F}}(G) \subseteq \text{Proj}_{\mathcal{F}}(G)$, it suffices to show that $\text{Proj}_{\mathcal{F}}(G) \subseteq \text{Cov}_{\mathcal{F}}(G)$ to complete the proof of the lemma.

Let $F$ be a member of $\text{Proj}_{\mathcal{F}}(G)$. We must show that $H = H^\mathcal{F}F$ whenever $F \leq H \leq G$. First of all, we observe that $G = G^\mathcal{F}F = G^\mathcal{F}H$. Hence, $H/H \cap G^\mathcal{F} \cong G/G^\mathcal{F} \in \mathcal{F}$, which indicates that $H^\mathcal{F} \leq G^\mathcal{F} \cap H$.

Let $K$ be a minimal supplement of $H^\mathcal{F}$ in $H$. Then $K \in \mathcal{F}$ by Lemma 2(1), and $G = HG^\mathcal{F} = (KH^\mathcal{F})G^\mathcal{F} = KG^\mathcal{F}$. We thus can find a minimal supplement $Y \leq K$ to $G^\mathcal{F}$ in $G$. By hypothesis, $Y$ belongs to $\text{Proj}_{\mathcal{F}}(G)$. So, $K = Y$, and it follows that $K$ is a minimal supplement of $G^\mathcal{F}$ in $G$, which shows $K \cap G^\mathcal{F} \leq \Phi(K)$.

We now claim that $\Phi(K)H^\mathcal{F}/H^\mathcal{F} \subseteq \Phi(H/H^\mathcal{F})$. Indeed, if $H = H^\mathcal{F}$ then $K = 1$ and the claim is trivial. Assume that $H^\mathcal{F} < H$. Let $M$ be any maximal subgroup of $H$ such that $M$ contains $H^\mathcal{F}$. Put $K_0 = K \cap M$. Then $M = M \cap H^\mathcal{F}K = H^\mathcal{F}(M \cap K) = H^\mathcal{F}K_0$. If $K_0 \triangleleft K_1 < K$ for some subgroup $K_1$ of $H$ then $M = K_1H^\mathcal{F}$, and so $K_1 \triangleq K_1 \cap K_0H^\mathcal{F} = K_0\langle K_1 \cap H^\mathcal{F} \rangle \leq K \cap M = K_0$; a contradiction. We thus see that $K_0$ is maximal in $K$. So, $\Phi(K) \leq K_0 \leq M$, which shows that $\Phi(K)H^\mathcal{F}/H^\mathcal{F}$ lies in every maximal subgroup of $H/H^\mathcal{F}$. The claim holds.

Now, $H \cap G^\mathcal{F}/H^\mathcal{F} = (KH^\mathcal{F}) \cap G^\mathcal{F}/H^\mathcal{F}$

$$= H^\mathcal{F}(K \cap G^\mathcal{F})/H^\mathcal{F} \leq \Phi(K)H^\mathcal{F}/H^\mathcal{F} \leq \Phi(H/H^\mathcal{F}),$$

and so

$$H/H^\mathcal{F} = (FG^\mathcal{F})\cap H/H^\mathcal{F} = F(G^\mathcal{F} \cap H)/H^\mathcal{F}$$

$$= FH^\mathcal{F}/H^\mathcal{F} \cdot G^\mathcal{F} \cap H/H^\mathcal{F} \leq FH^\mathcal{F}/H^\mathcal{F} \cdot \Phi(H/H^\mathcal{F}) \leq H/H^\mathcal{F},$$

which yields $H = FH^\mathcal{F}$ as desired. The proof is now complete.

**Lemma 4.** Let $\mathcal{F}$ be a subgroup-closed saturated formation. Then the following are equivalent:

(1) $G \in \mathcal{F}_{pc}$;

(2) for every subgroup $X \leq G$, $X$ is $\mathcal{F}$-subabnormal in $S_G(X)$;

(3) for every $\mathcal{F}$-subgroup $F \subseteq G$, $F$ is $\mathcal{F}$-subabnormal in $S_G(F)$.

**Proof.** (1) $\Rightarrow$ (2): If $G \in \mathcal{F}_{pc}$ then (2) holds by definition.