APPLICATIONS OF P-ADIC GENERALIZED FUNCTIONS AND APPROXIMATIONS BY A SYSTEM OF P-ADIC TRANSLATIONS OF A FUNCTION

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Abstract: Under some conditions we prove that the convergence of a sequence of functions in the space of P-adic generalized functions is equivalent to its convergence in the space of locally integrable functions. Some analogs are established of the Wiener tauberian theorem and the Wiener theorem on denseness of translations for P-adic convolutions and translations.

Keywords: P-adic generalized function, $L^p_{\text{loc}}(\mathbb{R}_+)$, multiplicative Fourier transform, Lebesgue points of order $p$, Wiener tauberian theorem, Wiener theorem on denseness of translations

Introduction

The classical theory of generalized functions originated from the books by S. L. Sobolev [1] and L. Schwartz [2]. Generalized functions were defined as linear functionals over various spaces of infinitely differentiable functions. In the case of dyadic derivatives a similar idea was developed by B. I. Golubov [3]. We use the notions of test and generalized functions on $\mathbb{R}_+$ similar to those in the book [4] by V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov in the case of the field of $p$-adic numbers.

We list the necessary definitions. Assume that $P = \{p_j\}_{j \in \mathbb{N}}$, with $p_j \in \mathbb{N}$ and $2 \leq p_j \leq N$ for all $|j| \in \mathbb{N}$. Put $m_0 = 1$, $m_j = p_1 \cdots p_j$, and $m_{-j} = 1/m_j$ for $j \in \mathbb{N}$. In this case, to every $x \in \mathbb{R}_+$ there corresponds the decomposition

$$
x = \sum_{j=1}^{k(x)} x_j m_{j-1} + \sum_{j=1}^{\infty} \frac{x_j}{m_j}, \quad x_j \in \mathbb{Z}, \quad 0 \leq x_j < p_j.
$$

(1)

The representation (1) is unique if we choose the decomposition with finitely many $x_j \neq 0$ in the case of $x = k/m_n$ ($k, n \in \mathbb{N}$). If $x, y \in \mathbb{R}_+$ are written in the form (1) then we define

$$
z = x \oplus y = \sum_{j=1}^{\max(k(x), k(y))} z_j m_{j-1} + \sum_{j=1}^{\infty} \frac{z_j}{m_j},
$$

where $z_j = x_j + y_j \pmod{p_j}$, $0 \leq z_j < p_j$, $|j| \in \mathbb{N}$. The quantity $x \ominus y$ is defined by analogy. A function $f : \mathbb{R}_+ \to \mathbb{C}$ is called $P$-continuous at $x$ whenever $\lim_{h \to 0} |f(x \oplus h) - f(x)| = 0$. Given $x, y \in \mathbb{R}_+$ of the form (1), we can write the kernel

$$
\chi(x, y) = \exp \left(2\pi i \sum_{j=1}^{\infty} (x_j y_j + x_j y_{-j}) \right)
$$

(the sum on the right-hand side is finite). The definition implies that $\chi(x, y) = \chi(y, x)$ and $|\chi(x, y)| = 1$ for $x, y \in \mathbb{R}_+$. Moreover (see [5, Section 1.5]), we have

$$
\chi(x \ominus z, y) = \chi(x, y) \chi(z, y), \quad \chi(x \ominus z, y) = \chi(x, y) \chi(z, y)
$$

for almost all pairs $(x, z) \in \mathbb{R}_+ \times \mathbb{R}_+$ and a fixed $y \in \mathbb{R}_+$. The function $D_y(x) = \int_0^y \chi(x, t) \, dt$ is an analog of the Dirichlet kernel. It is known that

$$
D_{m_n}(x) = m_n X_{[0, 1/m_n]}(x), \quad n \in \mathbb{Z},
$$

with $X_E$ the indicator function of $E$ (see [5, Section 1.5, 11.1]).
Put $I^n_j = [j/m_n, (j + 1)/m_n)$, $B_n = [0, m_n)$ (\(n \in \mathbb{Z}\) and \(j \in \mathbb{Z}_+\)), and let $G_{nm}$ designate the set of functions constant on all $I^n_j$ \((j \in \mathbb{Z}_+)\) and vanishing outside $B_n$. Consider $D = \bigcup_{n,m \in \mathbb{Z}} G_{nm}$. The sequence \(\{\varphi_k\}_{k=1}^\infty\) converges to $\varphi \in D$ in $D$ whenever

1. there exist $n, m \in \mathbb{Z}$ such that $\varphi_k \in G_{nm}$ for each $k \in \mathbb{N}$;
2. $\varphi_k(x)$ converges to $\varphi(x)$ uniformly on $\mathbb{R}_+$.

Clearly, $D$ is complete under this convergence; i.e., if the sequence \(\{\varphi_k\}_{k=1}^\infty \subset D\) satisfies the condition (1) and is uniformly fundamental on $\mathbb{R}_+$ then the limit function $\varphi$ lies in $D$. Furthermore, $D$ is dense in all spaces $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, endowed with the norm $\|f\|_p = (\int_{\mathbb{R}_+} |f(x)|^p \, dx)^{1/p}$. A simple consequence of the results by A. V. Efimov \([5, \text{Section 10.5, Theorem 10.5.2}]\) is the relation $\lim_{h \to 0} \|f(x + h) - f(x)\|_p = 0$, where $f \in L^p(\mathbb{R}_+)$, $1 \leq p < \infty$. The $L^\infty(\mathbb{R}_+)$ space consists of functions $f(x)$ measurable on $\mathbb{R}_+$ and such that $\|f\|_\infty = \sup_{x \in \mathbb{R}_+} |f(x)| < \infty$. The $L^p(B_1)$ spaces are defined by analogy.

Let $D'$ be the set of linear functionals $f$ over $D$. By $(f, \varphi)$ we denote the value of $f \in D'$ at $\varphi \in D$. We will write $f = g$ in $D'$ if $(f, \varphi) = (g, \varphi)$ for all $\varphi \in D$. Similarly \([4, \text{Section 6, Subsection 3}]\) it can be proven that $f \in D'$ is continuous, i.e., $\varphi_k \to \varphi$ in $D$ implies that $(f, \varphi_k) \to (f, \varphi)$. By definition, $f_n \to f$ in $D'$ if $\lim_{n \to \infty} (f_n, \varphi) = (f, \varphi)$ for each $\varphi \in D$. If $f \in L^p(B_n)$ for every $n \in \mathbb{N}$ then $f \in L^p_{\text{loc}}(\mathbb{R}_+)$. The sequence \(\{f_n\}_{n=1}^\infty\) converges to $f$ in $L^p_{\text{loc}}(\mathbb{R}_+)$ whenever $\lim_{n \to \infty} \|f - f_n\|_{L^p(B_n)} = 0$ for every $k \in \mathbb{N}$. Obviously, every function $f \in L^p_{\text{loc}}(\mathbb{R}_+)$ defines the functional in $D'$ by the formula $(f, \varphi) = \int_{\mathbb{R}_+} f(x) \varphi(x) \, dx$.

For $f \in D'$ and $\varphi \in D$, the functionals $f \varphi = f \varphi \in D'$ are defined by the equality $(f \varphi, \psi) = (f, \varphi \psi)$ for all $\psi \in D$. In this definition $\varphi \in D$ can be replaced with a function $\varphi$ constant on all $I^n_j$ for a fixed $n$. If for $f \in D'$ there exists $k \in \mathbb{Z}$ such that $f \chi_{B_k} = f$ then we write $\text{supp}(f) \subset B_k$. For $f \in L^q(\mathbb{R}_+)$ \((1 \leq q \leq \infty)\) and $g \in L^1(\mathbb{R}_+)$ the convolution $f \ast g(x)$ is defined as follows:

$$f \ast g(x) = \int_{\mathbb{R}_+} f(x \ominus y) g(y) \, dy.$$  

The Fubini and Riesz–Thorin theorems ensure that $f \ast g(x) \in L^q(\mathbb{R}_+)$ and $\|f \ast g\|_q \leq \|f\|_p \|g\|_1$. For $q = \infty$, the convolution $f \ast g$ is uniformly $P$-continuous on $\mathbb{R}_+$. Indeed, as was pointed above,

$$\lim_{h \to 0} \|g(x \ominus h) - g(x)\|_1 = 0$$

provided that $g \in L^1(\mathbb{R}_+)$. Hence, we have

$$|f \ast g(x \ominus h) - f \ast g(x)| \leq \|f\|_\infty \|g(\cdot \ominus h) - g(\cdot)\|_1 \to 0$$

uniformly on $x \in \mathbb{R}_+$ as $h \to 0$. Given $f \in D'$ and $\varphi \in D$, the convolution can be defined as $(f \ast \varphi, \psi) = (f, \tilde{\varphi} \ast \psi)$, where $\psi \in D$ and $\tilde{\varphi}(x) = \varphi(\ominus x)$. Since $\tilde{\varphi}, \varphi \ast \psi \in D$ for $\varphi, \psi \in D$ (see below), this definition is correct.

An important role for test and generalized functions is played by the convolution $S_m(f) = f \ast D_m$. For $f \in L^1(\mathbb{R}_+)$ the multiplicative Fourier $P$-transform is given by the formula

$$\hat{f}(x) = \int_0^\infty f(y) \chi(x, y) \, dy.$$  

Obviously, if $f_n(t)$ tends to $f(t)$ in $L^1(\mathbb{R}_+)$ then $\hat{f}_n(x)$ tends to $\hat{f}(x)$ uniformly on $\mathbb{R}_+$ and $\hat{f}(x)$ is $P$-continuous on $\mathbb{R}_+$ \(\text{see}[5, \text{Section 6.1, Theorem 6.1.5}]\). As is known \(\text{see}[5, \text{Section 6.2, Theorems 6.2.13 and 6.2.14}]\), $\hat{\varphi} \in G_{nm}$ for $\varphi \in G_{nm}$. Moreover \(\text{see}[5, \text{Section 6.2, Theorem 6.2.2}]\), since $\varphi \in D$ is $P$-continuous on $\mathbb{R}_+$, the inversion theorem

$$\varphi(x) = (\hat{\varphi})^\vee(x) := \int_0^\infty \hat{\varphi}(y) \chi(x, y) \, dy.$$