GLOBAL SOLVABILITY OF THE MULTIDIMENSIONAL
NAVIER–STOKES EQUATIONS OF A COMPRESSIBLE
FLUID WITH NONLINEAR VISCOSITY. I

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UDC 517.953

Introduction

We consider the system of equations of motion of a viscous compressible fluid which has the form
[1, 2]:
\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) = 0, \tag{0.1}
\]
\[
\frac{\partial (\rho \vec{u})}{\partial t} + \text{div}(\rho \vec{u} \otimes \vec{u}) = \text{div} \mathbf{P}' + \rho \vec{f}. \tag{0.2}
\]

Here \( \rho \) is the density of the fluid (a nonnegative function), \( \vec{u} \) is the velocity vector, \( \vec{f} \) is the given vector of external mass forces, \( \mathbf{P}' \) is the stress tensor (a given function of \( \rho \) and \( \vec{u} \)), \text{div} is the divergence in the space variables \( z \), and \( t \) is time.

The Stokes axioms of motion of a fluid [3] imply the representation

\[
\mathbf{P}' = \sum_{k=0}^{n-1} \alpha_k(\rho, J_s(\mathbf{D}))\mathbf{D}^k, \tag{0.3}
\]

where \( \mathbf{D} \) is the strain tensor with components \( D_{ij} = \frac{1}{2} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \); \( J_s \) are its fundamental invariants, \( s = 1, 2, \ldots, n \); \( \alpha_k \) are scalar coefficients, and \( n \in \mathbb{N} \) is the number of the space variables. In the literature, the classical model of a fluid is usually considered which corresponds to the simplest form (linear in \( \vec{u} \)) of the constitutive equation (0.3):

\[
\mathbf{P}' = (\lambda \text{div} \vec{u} - p(\rho))\mathbf{I} + 2\mu\mathbf{D}. \tag{0.3'}
\]

with the constant viscosity coefficients \( \lambda \) and \( \mu \) and a given function \( p(\rho) \) of the density which is referred to as pressure. The solvability theory of the equations (0.1) and (0.2) depends essentially on whether we consider the case of one, two, or three (and more) space variables. The one-dimensional case (more general models inclusive) is studied rather completely: there are theorems on global existence, uniqueness, stabilization of solutions, etc. [4–6]. In the multidimensional case the classical model (0.1), (0.2), (0.3') is not corroborated from the viewpoint of global solvability. The solutions constructed in [7] turned out to be essentially local as it was demonstrated in [8]. In this connection, only the refusal of the assumption of constancy of \( \lambda \) made it possible to establish a rather wide system of a priori estimates and prove existence of time-global weak, strong, and classical solutions to the two-dimensional problem [9]. As a matter of fact, in this article the solutions are constructed in Orlicz spaces. Seemingly, these spaces were used in the Navier–Stokes problem for the first time in [10], wherein an attempt was made at solving the two-dimensional problem; in this connection, we should also mention the article [11]. However, the time-global existence (for \( n = 2 \)) was first established in [9]. Up to that time there were known only uniqueness of classical solutions [12], time-local solvability [7,13–16], global existence in the case when the initial data are close to the rest state [17], and well-posedness of some approximate multidimensional models [18–20].
The aim of the present article is to prove solvability of the equations (0.1) and (0.2) for every \( n \geq 2 \). It is likely that this is possible only in the framework of the model of a nonlinear fluid, i.e., in the case when the equation (0.3) is nonlinear in \( \bar{u} \). The first step in this direction was made in [21], wherein the so-called Burgers model was considered, i.e., the situation in which \( \mathbb{P} \) depends only on \( \bar{u} \). It is curious to observe that the solvability problem for the Navier–Stokes equations of an incompressible fluid also takes various forms depending on whether we consider the one-, two-, or three-dimensional problem. In the last case it is only the nonlinearity of the constitutive equation for the stress that made it possible to prove global existence [22]. In many respects we follow [22]. Namely, we consider the case in which the density enters (0.3) only via the pressure which is a linear function of the former:

\[
\mathbb{P}' = -\rho I + \mathbb{P}(\bar{u}),
\]

and the tensor \( \mathbb{P}(\bar{u}) \) represents an arbitrary operator (in general, nonlocal in \( x \)) of \( \bar{u} \) which satisfies only the following axioms:

1. \( \mathbb{P} \) is coercive; i.e.,
   \[
   \int_{\Omega} \mathbb{P}(\bar{u}) : D(\bar{u}) \, dx \geq \int_{\Omega} M(\|D(\bar{u})\|) \, dx
   \]
   for all \( \bar{u} \in X \); here and in the sequel, we use the notation \( |B|^2 = B : B \) for tensors;
2. \( \mathbb{P} \) is monotone; i.e.,
   \[
   \int_{\Omega} (\mathbb{P}(\bar{u}) - \mathbb{P}(\bar{v})) : D(\bar{u} - \bar{v}) \, dx \geq 0 \text{ for all } \bar{u}, \bar{v} \in X;
   \]
3. \( \mathbb{P}(\cdot) \) acts boundedly and weakly continuously from \( X \) into \( L_{\overline{M}}(\Omega) \); i.e.,
   \[
   \int_{\Omega} \overline{M}(\|\mathbb{P}(\bar{u})\|) \, dx \leq C \left( 1 + \int_{\Omega} M(\|D(\bar{u})\|) \, dx \right)
   \]
   and \( \mathbb{P}(\bar{u} - \epsilon \bar{v}) \rightharpoonup \mathbb{P}(\bar{u}) \) weakly* in \( L_{\overline{M}}(\Omega) \) as \( \epsilon \to 0 \) for all \( \bar{u}, \bar{v} \in X \);
4. (optional) \( \int_{\Omega} \mathbb{P}(\bar{u}) : D(\bar{u}) \, dx \) is a convex functional of \( \bar{u} \).

Here we introduce the following notations: \( M(\cdot) \) is an \( N \)-function, \( \overline{M} \) is the complementary function, and

\[
X = \{ \bar{u} \mid D(\bar{u}) \in L_{\overline{M}}(\Omega); \bar{u}|_{\partial \Omega} = 0 \}, \quad \|\bar{u}\|_X = \|D(\bar{u})\|_{L_{\overline{M}}(\Omega)}.
\]

The notion of \( N \)-function, the definition of the Orlicz spaces \( L_M \) and \( E_M \), and the Orlicz classes \( K_M \) can be found in the monograph [23] (to within notations: in [23] the Orlicz space is denoted by \( L^\Phi \) and the Orlicz class by \( L^\Phi \)). We borrow the properties of the Sobolev-Orlicz spaces from [24] and the notion of the negative Sobolev-Orlicz spaces from [21]. Closing the introduction, we give a brief list of the basic definitions and properties of the above-mentioned spaces for the reader's convenience.

We confine exposition to the case in which

\[
M(s) \geq \exp(s), \quad s \geq s_0 = \text{const}.
\]

As an example, satisfying the axioms (1)–(3) ((1)–(4)), in which (0.4) agrees with (0.3) we can consider the constitutive equation of the form

\[
\mathbb{P}(\bar{u}) = \gamma_0 (\text{div } \bar{u}) I + \sum_{s=1}^{N} \gamma_s (\|D^s(\bar{u})\|^2) D^{2s-1}(\bar{u}),
\]

where \( \gamma_s \) is a monotone increasing continuous function\(^1\); moreover, \( M(\xi) = \gamma_1 (\xi^2)\xi^2 \) is an \( N \)-function satisfying the \( \Delta_3 \)-condition, \( \text{sgn} \gamma_0(s) = \text{sgn} s \), the functions \( \gamma_s \) for \( s \neq 1 \) grow essentially slower than \( \gamma_1 \), and \( N \) is an arbitrary natural number. In this case monotonicity of \( \mathbb{P} \) follows from Lemma 2.2, coerciveness is obvious, and finally the \( \Delta_3 \)-condition for \( M \) (in the axiom 3) implies that \( \overline{M}(M(s)) \sim \)

\(^1\) As mentioned, the arguments of the functions \( \gamma \) may also enter (0.7) nonlocally in \( x \), for instance, as integrals over \( \Omega \).