Abstract. A hybrid logic is obtained by adding to an ordinary modal logic further expressive power in the form of a second sort of propositional symbols called nominals and by adding so-called satisfaction operators. In this paper we consider hybridized versions of $S5$ ("the logic of everywhere") and the modal logic of inequality ("the logic of elsewhere"). We give natural deduction systems for the logics and we prove functional completeness results.

Keywords: Modal logic, hybrid logic, natural deduction, functional completeness

1. Introduction

We shall begin this paper by making a few introductory remarks on the topics of hybrid logic, natural deduction, and functional completeness.

Hybrid logic is obtained by adding to ordinary multi-modal logic further expressive power in the form of a second sort of propositional symbols called nominals, and moreover, by adding so-called satisfaction operators. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. Thus, in hybrid logic a name is a particular sort of propositional symbol whereas in first-order logic it is an argument to a predicate. If $a$ is a nominal and $\phi$ is an arbitrary formula, then a new formula $a : \phi$ called a satisfaction statement can be formed. The part $a$ of $a : \phi$ is called a satisfaction operator. The satisfaction statement $a : \phi$ expresses that the formula $\phi$ is true at one particular world, namely the world at which the nominal $a$ is true. We shall consider hybridized versions of $S5$ and the modal logic of inequality. They are both mono-modal logics. In the former case a modal operator, $\Box$, is interpreted using the universal relation (wherefore we have called it "the logic of everywhere") and in the latter case a modal operator, $\mathcal{R}$, is interpreted using the relation of inequality (wherefore it is called "the logic of elsewhere" cf. Krister Segerberg’s paper [33]). The former logic will be denoted $\mathcal{H}(\Box)$ and the latter one $\mathcal{H}(\mathcal{R})$.

1 The fact that the relations involved in the logics are the universal relation and the relation of inequality makes it tempting to draw a parallel to Alfred Tarski’s paper [36]

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The history of hybrid logic goes back to Arthur Prior’s work, more precisely, it goes back to what he called four grades of tense logical involvement. They were presented in the book [32], Chapter XI (also Chapter XI in the new edition [22]). See also [31] Chapter V.6 and Appendix B.3–4. Prior introduced nominals and satisfaction operators, and moreover, he introduced the so-called binder $\forall$ which is analogous to the standard first-order universal quantifier. We shall not consider binders in the present paper, however. See [27] for an account of Prior’s work. The paper [7] makes a connection to Donald Davidson’s notion of a theory of truth.

Now, natural deduction style inference rules for ordinary classical first-order logic were originally introduced by Gerhard Gentzen in [20] and later on considered by Dag Prawitz in [28, 29]. With reference to Gentzen’s work, Prawitz made the following remarks on the significance of natural deduction.

...the essential logical content of intuitive logical operations that can be formulated in the languages considered can be understood as composed of the atomic inferences isolated by Gentzen. It is in this sense that we may understand the terminology natural deduction.

Nevertheless, Gentzen’s systems are also natural in the more superficial sense of corresponding rather well to informal practices; in other words, the structure of informal proofs are often preserved rather well when formalised within the systems of natural deduction. ([29], p. 245)

Natural deduction systems are characterised by having two different kinds of rules for each non-nullary connective; there is a kind of rules which introduces a connective and there is a kind of rules which eliminates a connective. A maximum formula in a derivation is then a formula occurrence that is both introduced by an introduction rule and eliminated by an elimination rule. A maximum formula can be considered a “detour” in the derivation and it can be removed by rewriting the derivation. Such a step is called a reduction step and it is a notable feature of natural deduction systems that they satisfy a so-called normalisation theorem which says that any derivation can be rewritten to a derivation without maximum formulas by repeated applications of reductions. Another notable feature of natural deduction

where he argues that given an arbitrary set, there are only four binary relations on the set which should be called “logical”, namely the empty relation, the universal relation, the relation of equality, and the relation of inequality. Tarski’s argument is based on the observation that these relations are exactly the binary relations on the set which are mapped to themselves by all bijections on the set in question.