Abstract. A logic is selfextensional if its interderivability (or mutual consequence) relation is a congruence relation on the algebra of formulas. In the paper we characterize the selfextensional logics with a conjunction as the logics that can be defined using the semilattice order induced by the interpretation of the conjunction in the algebras of their algebraic counterpart. Using the characterization we provide simpler proofs of several results on selfextensional logics with a conjunction obtained in [13] using Gentzen systems. We also obtain some results on Fregean logics with conjunction.

Keywords: algebraic logic, selfextensional logic, Fregean logic, algebraizable logic, generalized matrix, full generalized model, fully adequate Gentzen system.

1. Introduction

The concept of logic that is taken as primary in Abstract Algebraic Logic (AAL) is that of a consequence relation between sets of formulas and formulas which has the substitution-invariance property; this means that if a formula $\varphi$ is a consequence (according to the logic) of a set of formulas $\Gamma$, then for every set of formulas $\Delta$ and every formula $\psi$ that are obtained from $\Gamma$ and $\varphi$ by replacing simultaneously some sentential variables by formulas, the formula $\psi$ is also a consequence of $\Delta$. A logic $S$ in this sense may have different replacement properties. The strongest one, shared for example by classical, intuitionistic and all the intermediate propositional logics, says that if $\Gamma, \varphi \vdash_S \psi$ and $\Gamma, \psi \vdash_S \varphi$, then

$$\Gamma, \delta(p/\varphi) \vdash_S \delta(p/\psi) \text{ and } \Gamma, \delta(p/\varphi) \vdash_S \delta(p/\psi),$$

where $\delta(p/\varphi)$ and $\delta(p/\psi)$ are the formulas obtained by substituting respectively $\varphi$ and $\psi$ for $p$ in $\delta$. This replacement property is a formal counterpart of Frege's compositionality principle for truth. Logics satisfying this replacement property are studied in [10] where they are called Fregean; the origin of

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the name comes from R. Suszko’s studies [32, 33] on his non-Fregean logic. Many important logics are not Fregean: for instance, almost all the logics of the modal family. Some logics of this family, like the local consequence relations of the normal modal logics, satisfy a weaker replacement property: for all formulas $\varphi$, $\psi$, $\delta$,

if $\varphi \vdash_S \psi$ and $\psi \vdash_S \varphi$, then $\delta(p/\varphi) \vdash_S \delta(p/\psi)$ and $\delta(p/\varphi) \vdash_S \delta(p/\psi)$.

A logic is said to be selfextensional if it satisfies this weaker replacement property. Using algebraic terminology this means that the mutual consequence relation between formulas is a congruence relation of the formula algebra. R. Wójcicki coined the name and started the study of this family of logics in [34].

One of the aims of AAL is to develop a general theory of the algebraization of logics that (1) defines and justifies on principled grounds which class of algebras should be considered the algebraic counterpart of a given logical system and (2) establishes with enough generality connections between the possible metalogical properties of a logical system and the possible algebraic properties of its algebraic counterpart. The pursuit of this aim has led to fruitful concepts which are used to classify the different logical systems according to the ties they have with their algebraic counterpart.

The classification obtained is known as the Leibniz hierarchy. In this hierarchy logics are divided first of all into protoalgebraic and non-protoalgebraic, and then the first class is subdivided (mainly) into regularly algebraizable, algebraizable, weakly algebraizable and equivalential logics. Protoalgebraic logics can be defined in several (equivalent) ways. The easiest one to state is to define them as the logics with a generalized form of implication with very weak properties, i.e. a set of formulas in two variables $[p \Rightarrow q]$ that satisfies the generalized modus ponens rule, or detachment rule, $(p, [p \Rightarrow q] \vdash q)$ and that for every $\varphi \in [p \Rightarrow q]$ the formula $\varphi(q/p)$ is a theorem. Protoalgebraic logics form a natural class for which the semantics of logical matrices behaves very well: many of the results of universal algebra have counterparts of specific logical interest in the theory of logical matrices for protoalgebraic logics. For information on AAL and the Leibniz hierarchy the reader is addressed to [7, 16].

We find selfextensional logics in every class of the Leibniz hierarchy. Many of the best-known selfextensional logics are protoalgebraic, for example classical logic, intuitionistic logic and the local consequences of normal modal logics. But as soon as we move to fragments of these logics we find selfextensional logics that are not protoalgebraic, for example the conjunction-disjunction fragment of classical logic [18], the implication-less