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A Mixed $\lambda$-calculus

Abstract. The aim of this paper is to define a $\lambda$-calculus typed in a Mixed (commutative and non-commutative) Intuitionistic Linear Logic. The terms of such a calculus are the labelling of proofs of a linear intuitionistic mixed natural deduction $\text{NILL}$, which is based on the non-commutative linear multiplicative sequent calculus $\text{MNL}$ [RuetAbrusci 99]. This linear $\lambda$-calculus involves three linear arrows: two directional arrows and a non-directional one (the usual linear arrow). Moreover, the $\lambda$-terms are provided with series-parallel orders on free variables.

We prove a normalization theorem which explicitly gives the behaviour of the order during the normalization procedure.

Keywords: Typed $\lambda$-calculus, non-commutative linear logic, order varieties, series-parallel orders, normalization.

1. Introduction

For more than a decade, some significant studies have focused on Linear Logic in a mixed (commutative and non-commutative) setting. P. de Groote defined the partially commutative linear calculus [de Groote 96], that is a linear intuitionistic sequent calculus with two versions (commutative and non-commutative) of multiplicative connectives. M.V. Abrusci and P. Ruet [RuetAbrusci 99] introduced proof nets and a sequent calculus for the multiplicative fragment of mixed (commutative and non-commutative) linear logic. This latter study work was then extended by Ruet to full linear logic [Ruet 2000]. Several other works have, until now, been developed in various and fruitful directions.

Let us recall that the works quoted above can be seen as extensions of the Lambek Calculus [Lambek 58] which corresponds to an intuitionistic, strictly non-commutative, multiplicative fragment of Linear Logic. The Lambek calculus was defined in order to formalize the natural language grammar. Note that some interesting works with linguistic motivations have been proposed using the $\lambda$-calculus framework. Some authors studied the $\lambda$-calculus in a non commutative setting: a distinction between a left-searching and a right-searching $\lambda$-abstracter has been suggested by W. Buszkowski in the late
1980s [Buszkowski 1987], [Buszkowski 1988]; Lambda-calculi with such \(\lambda\)-operators have then been investigated by H. Wansing in the early 1990s [Wansing 92], [Wansing 93]. More recently, R. Muskens and P. de Groote independently defined grammars having a fine syntax/semantics interface; in these two grammatical frameworks, the syntactic information is represented using (usual) typed \(\lambda\)-terms [de Groote 2001], [Muskens 2001]. We suggest that a \(\lambda\)-calculus typed in a mixed logic (with commutative and non-commutative connectives) could be an interesting tool for the formalisation of natural languages. This was one of the reasons why we decided to develop such a mixed \(\lambda\)-calculus. Another one was to apply proof-theorical tools in a mixed setting.

The aim of this paper is thus to define a \(\lambda\)-calculus typed in a Mixed Linear Logic. The steps to achieving this are:

– to define a mixed linear intuitionistic natural deduction. The \(\lambda\)-terms can then be obtained by labelling the proofs of such a natural deduction system. Note that, as a consequence of selecting a mixed setting, the linear \(\lambda\)-calculus that we define includes three linear arrows: two directional arrows and a non-directional one (the usual linear arrow). Moreover in the two known mixed systems ([RuetAbrusci 99] and [de Groote 96]), sequents are sets of formula occurrences provided with some order notion. in the same way, the \(\lambda\)-terms we deal with are provided with some order on the set of free variables.

– to prove the normalization in the \(\lambda\)-calculus by checking the termination and confluence. This result is explicitly obtained within our framework by calculating the effect of normalization on the order. This is the real unique aspect of our result.

Such a system of mixed natural deduction system undoubtedly could be defined starting from the partially commutative intuitionistic and linear sequent calculus of P. de Groote [de Groote 96]. We preferred to set our work within a framework allowing us to more easily come back to a symmetrical framework and in which structural rules do not have to be added, in particular we do not need any entropy rule. Nevertheless, the mixed linear intuitionistic natural deduction system that we define (denoted by \(NILL\)) is equivalent to the implicatve fragment of the de Groote system regarding provability. The proof of this equivalence is given in appendix B.

We based our definition of the system deduction \(NILL\) on the sequent calculus \(MNL\) introduced by P. Ruet and M. Abrusci in [RuetAbrusci 99]. In \(MNL\), the one-side sequents \(\vdash \Gamma\) are the data of a set \(\Gamma\) of formula