Abstract. Action logic of Pratt [21] can be presented as Full Lambek Calculus FL [14, 17] enriched with Kleene star \(*\); it is equivalent to the equational theory of residuated Kleene algebras (lattices). Some results on axiom systems, complexity and models of this logic were obtained in [4, 3, 18]. Here we prove a stronger form of \(*\)-elimination for the logic of \(*\)-continuous action lattices and the \(\Pi^0_1\)–completeness of the equational theories of action lattices of subsets of a finite monoid and action lattices of binary relations on a finite universe. We also discuss possible applications in linguistics.

Keywords: Kleene algebra, action algebra, relation algebra, categorial grammar.

1. Introduction

Kleene algebras are idempotent semirings with Kleene star \(*\) (Kozen [10, 11]). Action algebras are residuated Kleene algebras. They were introduced by Pratt [21] to provide an equational axiomatization of the equations valid in Kleene algebras; by the Kozen completeness theorem [11], these equations are precisely the equations true for regular expressions. Action algebras are algebraic models of Full Lambek Calculus (FL) with \(*\) but without \(\land\). With \(\land\), they are called action lattices. Concrete action algebras and lattices appear in mathematical linguistics (algebras of languages) and logics of programs (algebras of relations).

Action Logic (ACT) is a propositional logic, corresponding to action algebras (equivalent to the equational theory of action algebras). It is not known whether ACT admits a cut-free sequent axiomatization and is decidable [9].

A Kleene algebra is \(*\)-continuous, if \(xa^*y = \bigvee\{xa^n y : n \in \omega\}\), for all elements \(x, a, y\). Algebras of languages and algebras of relations are \(*\)-continuous. From the Kozen completeness theorem it follows that the equational theory of all Kleene algebras is decidable and equals that of \(*\)-continuous Kleene algebras; they amount to the equational theory of relational Kleene algebras [13].

In [4, 3], it has been shown that the situation is different for the case of action algebras (lattices). The equational theory of \(*\)-continuous action
algebras (lattices) is \( \Pi^0_1 \)-complete, whence it strictly contains the equational theory of all action algebras (lattices) which is \( \Sigma^0_1 \). The former possesses Finite Model Property (FMP), but the latter does not (otherwise, it would be equal to the former, by the \(*\)-continuity of finite action algebras). The equational theory of Kleene algebras possesses FMP [19]. The equational theory of action algebras of relations is \( \Pi^0_1 \)-hard. The equational theory of algebras of regular languages is \( \Pi^0_1 \)-complete. The equational theory of algebras of languages is not in \( \Sigma^0_1 \cup \Pi^0_1 \).

The logic of \(*\)-continuous action lattices (\( ACT_\omega \)) is an infinitary logic: an extension of FL by some rules for \( * \), and one of them is an \( \omega \)-rule [4]. The cut-elimination theorem and a theorem on \(*\)-elimination for \( ACT_\omega \) are proved in [18]. As a consequence of the latter theorem, \( ACT_\omega \) is \( \Pi^0_1 \). It is \( \Pi^0_1 \)-hard, since the total language problem for context-free grammars is reducible to it, using categorial grammars and cut-elimination for FL [4]. The same reduction yields the \( \Pi^0_1 \)-hardness of other theories of action algebras, mentioned above.

The present paper continues this research. We obtain some new results and discuss certain applications of action logic.

In section 2, we define basic algebraic notions and present the infinitary system \( ACT_\omega \). The \(*\)-elimination theorem from [18] is strengthened: we eliminate all (not only negative) occurrences of \( * \). We also sketch a different proof of the elimination of negative occurrences of \( * \). In section 3, we use the reduction, applied in [4] in the proof of \( \Pi^0_1 \)-hardness of \( ACT_\omega \), to prove the \( \Pi^0_1 \)-completeness of the equational theories of powerset action lattices over finite monoids and finite relational (square) action lattices. We briefly announce other results of that kind. In section 4, we discuss certain problems, connected with applications of action logic in categorial grammars.

Theorems 4 and 5 are common results of both authors; they have been included in PhD Thesis of the second author [19] with slightly different proofs. The new proof of theorem 1 and the remaining results of this paper are due to the first author.

2. Action logic and *-elimination

A Kleene algebra is an algebra \( M = (M, \vee, \cdot, *, 0, 1) \) such that \( (M, \vee) \) is a (join) semilattice, 0 is its lower bound, \( (M, \cdot, 1) \) is a monoid, product \( \cdot \) distributes over \( \vee \), \( 0a = 0 = a0 \), and \( * \) is a unary operation, satisfying:

\[
1 \vee aa^* \leq a^*; \quad 1 \vee a^*a \leq a^*,
\]

if \( ab \leq b \) then \( a^*b \leq b \); if \( ba \leq b \) then \( ba^* \leq b \),

(1)

(2)