Abstract. Using the axiom system provided by Carsten Augat in [1], it is shown that the only 6-variable statement among the axioms of the axiom system for plane hyperbolic geometry (in Tarski’s language $L_{B\equiv}$), we had provided in [3], is superfluous. The resulting axiom system is the simplest possible one, in the sense that each axiom is a statement in prenex form about at most 5 points, and there is no axiom system consisting entirely of at most 4-variable statements.

Keywords: Hyperbolic geometry, Euclidean geometry, simplicity, axiom system, betweenness and equidistance.

1. Introduction

In [3] we have presented several axiom systems for plane hyperbolic and Euclidean geometry. Those that were expressed in constructive languages, i. e. in languages without relation symbols, were shown to be the simplest possible in the sense that each of their axioms, with one exception, an existential sentence with two individual variables, was a universal statement with at most four variables. In the case of the axiom systems for plane hyperbolic and Euclidean geometry that were expressed in Tarski’s one-sorted first-order language $L_{B\equiv}$ (with individual variables to be interpreted as points, and with two relation symbols, a ternary one, $B$, and a quaternary one $\equiv$, with $B(abc)$ to be read as ‘point $b$ lies between $a$ and $c$’, and $ab \equiv cd$ to be read as ‘segment $ab$ is congruent to segment $cd$’), we knew that the one for the Euclidean plane was not simplest possible — since it contained an axiom, A20, a universal statement about 6 variables, and we had already provided in [2] an axiom system for the same theory all of whose axioms are, when written in prenex form, statements about at most 5 variables — and we did not know whether there was an axiom system for plane hyperbolic geometry all of whose axioms are, when written in prenex form, statements about at most 5 variables. We had shown that any axiom system in $L_{B\equiv}$ for plane hyperbolic geometry had to have at least a 5-variable axiom, so the
possibility was left open that, just like for plane Euclidean geometry, there is an axiom system for plane hyperbolic geometry, all of whose axioms, written in prenex form, are statements about at most 5 variables, and that axiom system is the simplest possible one. The aim of this note is to point out that this is indeed true, since axiom A20 of [3] is superfluous. The reason why it is superfluous is contained in the doctoral dissertation of Carsten Augat [1].

2. Augat’s axiom system for metric planes with midpoints

We recall that to prove that the structures axiomatized by the axioms A1-A21 in [3] are metric planes, which was the most arduous part of the proof, we had used an axiom system for non-elliptic metric planes from [5], in which A20 occurred (although it is not known whether it is actually needed in that axiom system as well, as it might turn out to be superfluous).

C. Augat has presented in [1, p. 13–30] an axiom system for non-elliptic metric planes in which every point-pair has a midpoint. It can be expressed in \( L_{L^\equiv} \), where \( L \) is the collinearity predicate, with \( L(abc) \) to be read ‘the points \( a, b, \) and \( c \) are collinear’ (and \( a, b, c \) do not need to be distinct). The axioms are:

M 1. \( L(aba) \),

M 2. \( L(abc) \rightarrow L(cba) \land L(bac) \),

M 3. \( a \neq b \land L(abc) \land L(abd) \rightarrow L(acd) \),

M 4. \( ab \equiv cd \land ab \equiv ef \rightarrow cd \equiv ef \),

M 5. \( ab \equiv ba \),

M 6. \( ab \equiv cc \rightarrow a = b \),

M 7. \( (\forall ab)(\exists c) L(abc) \land ca \equiv cb \),

M 8. \( (\forall ab)(\exists c) a \neq b \rightarrow L(abc) \land ab \equiv ac \land b \neq c \),

M 9. \( p \neq q \land ap \equiv aq \land bp \equiv bq \rightarrow (cp \equiv cq \leftrightarrow L(abc)) \),

M 10. \( a \neq c \land b \neq d \land L(amc) \land ma \equiv mc \land L(bmd) \land mb \equiv md \rightarrow ad \equiv bc \),

M 11. \( (\forall abp)(\exists p') \neg L(abp) \rightarrow p \neq p' \land ap \equiv ap' \land bp \equiv bp' \),

M 12. \( (\exists abc) \neg L(abc) \).