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A Modal Logic for Mixed Strategies

Abstract. Modal logics have proven to be a very successful tool for reasoning about games. However, until now, although logics have been put forward for games in both normal form and games in extensive form, and for games with complete and incomplete information, the focus in the logic community has hitherto been on games with pure strategies. This paper is a first to widen the scope to logics for games that allow mixed strategies. We present a modal logic for games in normal form with mixed strategies, and demonstrate its soundness and strong completeness. Characteristic for our logic is a number of infinite rules.

Keywords: Modal Logic, Logics for Games, Mixed Strategies.

1. Introduction

The recent years have seen a flurry of activities relating modal logics to game theory. A survey is [8] although since its appearance several new developments have started. One of the early interests from modal logic in game theory was through the line of epistemic and doxastic logic, where game theorists formulated the epistemic conditions for Nash equilibrium in [2]. Notions like information partitions and events as used in [2] are easily seen to correspond to $S_5$ accessibility relations and possible worlds, respectively. As remarked in [8], the initial ideas from game theory that the logic community paid attention to were relatively uninteresting for the game theorist — mostly 2-player extensive games of perfect information which are strictly competitive. However, [8] then also observes that more recent work in logic has extended its scope considerably, introducing, e.g., cooperative game theory, imperfect information and games involving more than 2 players.

However, as far as we are aware, no matter how rich the modal frameworks, including dynamic notions of belief and incorporating ways to refer to the preferences of agents, have become, all the modal logical work for games until now has focussed on reasoning about games with pure strategies only. That is, the players chose their actions from a, usually finite, set, and

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they do not randomize over them. This is a serious limitation: one of the most celebrated theorems of game theory for instance holds that every finite strategic game has a mixed strategy Nash equilibrium (e.g., [6, p. 33]). This result is not true if one restricts himself to only considering pure strategies, as for instance witnessed by the matching pennies game ([6, Example 17.1]).

In this paper we introduce a modal logic, *Modal Logic for Mixed Strategies* (MLMS), in which one can reason about mixed strategies in games. MLMS is interpreted directly on a given set of mixed strategies of a game, and we provide a proof system that is both sound and strongly complete. The proof system for MLMS exploits some non-standard, infinite rules. They cater to the fact that although the sets of pure strategies for each player are assumed to be finite, allowing him to randomize over them presents him with an infinite number of choices. Throughout the paper, we show how to use the logics to reason about the matching pennies game.

Our logic MLMS, and in particular having a sound and complete axiomatisation, clarifies what is exactly needed in proving game-theoretic results (that are expressible in our language) concerning mixed strategies. It also enables one to compare logics for games with pure strategies with ours: MLMS for instance needs a rule with infinitely many premises, which seems inherently linked to the fact that we have mixed strategies. Moreover, linking a fragment of game theory to a well-established framework like modal logic, opens up a suite of possible tools, like theorem provers and model checkers. For instance, there are probabilistic model checkers (like PRISM, [5]) which allow for modal temporal languages, and might be adapted to verify properties involving mixed equilibria in a given game. Having said this, we should also add that our results are only preliminary and limited: our axiom system for instance suggests that one may need to look at sub-systems to get computationally feasible proof systems or model checkers.

## 2. Games and mixed strategies

Our work builds on the notion of a normal form game, or game for short, in this paper.

**Definition 2.1 (Game).** A *game* is a tuple

\[ G = \langle Ag, \{\Pi_i\}_{i \in Ag}, \{u_i\}_{i \in Ag} \rangle, \]

where \( Ag \) is a finite set of players, and for each agent \( i \in Ag \), we have a set \( \Pi_i \) of pure strategies. We assume an order on \( Ag \) and generally use the number in the order to refer to the agents. A pure strategy profile is a tuple