Abstract. One of the problems we face in many-valued logic is the difficulty of capturing the intuitive meaning of the connectives introduced through truth tables. At the same time, however, some logics have nice ways to capture the intended meaning of connectives easily, such as four-valued logic studied by Belnap and Dunn. Inspired by Dunn’s discovery, we first describe a mechanical procedure, in expansions of Belnap-Dunn logic, to obtain truth conditions in terms of the behavior of the Truth and the False, which gives us intuitive readings of connectives, out of truth tables. Then, we revisit the notion of functional completeness, which is one of the key notions in many-valued logic, in view of Dunn’s idea. More concretely, we introduce a generalized notion of functional completeness which naturally arises in the spirit of Dunn’s idea, and prove some fundamental results corresponding to the classical results proved by Post and Slupecki.

Keywords: Belnap-Dunn logic, Relational semantics, Functional completeness.

1. Introduction

1.1. Background and Motivation

After the birth of many-valued logic in 1920s realized independently by Jan Łukasiewicz and Emil Post, various kinds of such system have been developed based on different motivations. For example, Kurt Gödel proved a well-known result that there are no finite-valued semantics for intuitionistic logic.¹ Dmitri Bochvar introduced a three-valued logic with a motivation to deal with paradoxes, whereas Stephen Cole Kleene introduced another three-valued logic with a motivation in computability theory. These are the systems developed already in 1930s, and there are more and more many-valued logics developed in the literature.

¹ Note though that intuitionistic logic is complete with respect to infinite-valued semantics, as reported by Stanisław Jaśkowski in [20].

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Now, the systems we focus on in this paper are four-valued logic known as Belnap-Dunn logic\textsuperscript{2} (BD hereafter), and its expansions. BD was originally developed by Michael Dunn, and later stepped further to apply the logic to computer science by Nuel D. Belnap Jr. And expansions of BD we consider include systems of Kleene and Graham Priest. The motivation for Priest was to consider the problem of paradox, and to this end, he studied a three-valued logic now known as the Logic of Paradox.\textsuperscript{3}

Let us now observe an interesting aspect of BD. Originally, the semantics of BD was given as four-valued semantics, first discovered by Timothy Smiley, according to \cite{15, p. 428}. However, at that time, there were no semantic intuitions for the four values. And it was Dunn who gave the intuitive reading for the four values, namely four values being subsets of \{the True, the False\}. We will write the True and the False as 1 and 0, and following this notation, the four values will be $\emptyset, \{0\}, \{1\}$ and $\{0, 1\}$. Since these can be read as neither true nor false, false only, true only and both true and false, we take the capital letters of them, namely $n$, $f$, $t$ and $b$ respectively. Note, however, that the approaches of Belnap and Dunn to the above discovery of Dunn were different. Indeed, on the one hand, Belnap took the two ‘non-classical’ values $b$ and $n$ seriously, whereas Dunn tried to ‘deontologize’ those values by considering relations instead of functions. And this consideration by Dunn led to an appealing semantics, known as relational semantics.\textsuperscript{4} The main difference between these two semantics lies in the number of truth values. In the many-valued semantics, there are more than two truth values for non-classical logics whereas in relational semantics, there are only two truth values even for the semantics of the non-classical logics. Accordingly, the connectives of the system that are interpreted as functions in the many-valued semantics, are interpreted as relations in relational semantics. Briefly speaking, relational semantics is appealing, at least to the authors, since it serves as a convenient tool to grasp the intuitive meaning of connectives given by truth tables. We consider this tool to be important since there are many cases that we find difficult to read off the

\textsuperscript{2}This logic is also referred to as Dunn–Belnap logic (cf. \cite{19}), de Morgan logic (cf. \cite{9}) and First Degree Entailment (FDE, cf. \cite{28}). A nice introduction to FDE can be found in [28, chap. 8], and a brief historical remark can be found in [15, p. 428, footnote 2].

\textsuperscript{3}Note that the truth tables for this system was already realized by Asenjo in [3], but it was Priest who seriously took the system and proved various results on the system.

\textsuperscript{4}This should not be confused with the usage of relational semantics to refer to Kripke semantics for modal and other intensional logics where accessibility relations play an important role. Note also that there is a Kripke semantics for BD, discovered by Richard Routley and Val Routley. Details can be found, for example, in [28].