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The Weak Choice Principle WISC may Fail in the Category of Sets

Abstract. The set-theoretic axiom WISC states that for every set there is a *set* of surjections to it cofinal in *all* such surjections. By constructing an unbounded topos over the category of sets and using an extension of the internal logic of a topos due to Shulman, we show that WISC is independent of the rest of the axioms of the set theory given by a well-pointed topos. This also gives an example of a topos that is not a predicative topos as defined by van den Berg.

Keywords: WISC, Choice principle, Set theory, ETCS, Toposes.

1. Introduction

Well-known from algebra is the concept of a *projective object*: in a finitely complete category this is an object P such that any epimorphism with codomain P splits. The axiom of choice (AC) can be stated as saying that every set is projective in the category of sets. Various constructive set theories seek to weaken this, and in particular the axiom known as PAx (Presentation Axiom) [1] or CoSHEP (Category of Sets Has Enough Projectives) asks merely that every set X has an epimorphism $P \twoheadrightarrow X$ where P is a projective set. Many results that seem to rely on the axiom of choice, such as the existence of enough projectives in module categories, may be proved instead with PAx. As a link with a more well-known axiom, PAx implies the axiom of dependent choice.

There is, however, an even weaker option, here called WISC (to be explained momentarily). Consider the full subcategory $Surj/X \hookrightarrow \mathbf{set}/X$ of surjections with codomain X, in some category $\mathbf{set}$ of sets; clearly it is a large category. Then PAx implies the statement that $Surj/X$ has a *weakly initial object*, namely an object with a map to any other object, not necessarily unique (the axiom of choice says $\mathrm{id}_X : X \to X$ is weakly initial in

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$Surj/X$). Another way to think of the presentation axiom is that for every set X there is a 'cover' $P \twoheadrightarrow X$ such that any surjection $Y \twoheadrightarrow P$ splits.

The axiom WISC (Weakly Initial Set of Covers), due to Toby Bartels and Mike Shulman, asks merely that the category $Surj/X$ has a weakly initial *set*, for every X. This is a set I_X of objects (that is, of surjections to X) such that for any other object (surjection), there is a map from *some* object in I_X. To continue the geometric analogy, this is like asking that there is a set of covers of any X such that each surjection $Y \twoheadrightarrow X$ splits locally over at least one cover in that set. An example implication of WISC is that the cohomology $H^1(X, G)$ defined by Blass in [2] is indeed a set. The assertion that $H^1(X, G)$ is a proper class seems to be strictly weaker than ¬WISC, but to the author's knowledge no models have yet been produced where this is the case.

The origin of the axiom WISC (see [8]) was somewhat geometric in flavour but the question naturally arises whether toposes, and in particular the category of sets, can fail to satisfy WISC. A priori, there is no particular reason why WISC should hold, so the burden is to supply an example where it fails. It goes without saying that neither AC nor PAx can hold in such an example.

The first result in this direction was from van den Berg (see [12][1]) who proved that WISC implies the existence of a proper class of regular cardinals, and so WISC must fail in Gitik's model of ZF [4]. This model is constructed assuming the existence of a proper class of certain large cardinals, and it has no regular cardinals bigger than $\aleph_0$. Working in parallel to the early development of the current paper, Karagila [5] gave a model of ZF in which there is a proper class of incomparable sets (sets with no injective resp. surjective functions between them) surjecting onto the ordinal ω. This gave a large-cardinal-free proof that WISC was independent of the ZF axioms, answering a question raised by van den Berg.

The current paper started as an attempt to also give, via category-theoretic methods, a large-cardinals-free proof of the independence of WISC from ZF. Since the release of [5], this point is moot as far as independence from ZF goes. However, the proof in [5] relies on a symmetric submodel of a class-forcing model, which is rather heavy machinery. Thus this paper, while proving a slightly weaker result, does so with, in the opinion of the author, far less.

[1] In that paper, WISC is used in a guise of an equivalent axiom called AMC, the Axiom of Multiple Choice. To avoid confusion with other axioms with that name, this paper sticks with the term 'WISC'.