Abstract. We consider a family of logical systems for representing entailment relations of various kinds. This family has its root in the logic of first-degree entailment formulated as a binary consequence system, i.e. a proof system dealing with the expressions of the form $\varphi \vdash \psi$, where both $\varphi$ and $\psi$ are single formulas. We generalize this approach by constructing consequence systems that allow manipulating with sets of formulas, either to the right or left (or both) of the turnstile. In this way, it is possible to capture proof-theoretically not only the entailment relation of the standard four-valued Belnap’s logic, but also its dual version, as well as some of their interesting extensions. The proof systems we propose are, in a sense, of a hybrid Hilbert–Gentzen nature. We examine some important properties of these systems and establish their completeness with respect to the corresponding entailment relations.

Keywords: First-degree entailment, Belnap’s logic, Four-valued logic, Consequence system.

1. Preliminaries: First-Degree Entailment, a Consequence System and Four Truth Values

The notion of first-degree entailment was first put into circulation by Belnap in a short abstract of his talk at the twenty-fourth annual meeting of the Association for Symbolic Logic held on Monday, December 28, 1959 at Columbia University in New York [4]. It was defined there as an expression of the form $\varphi \rightarrow \psi$, where both $\varphi$ and $\psi$ are formulas containing only $\land$, $\lor$, $\neg$ (and maybe other truth-functional connectives defined by these). For a justification of first-degree entailments Belnap developed a machinery of “tautological entailments” conceived as a tool of their “validation”. Tautological entailments are essentially expressions of the form $\varphi_1 \lor \ldots \lor \varphi_m \rightarrow \psi_1 \land \ldots \land \psi_n$ (or reducible to them by special replacement rules), where every $\varphi_i \rightarrow \psi_j$ is in its turn of the form $\chi_1 \land \ldots \land \chi_m \rightarrow \xi_1 \lor \ldots \lor \xi_n$, where $\chi_1, \ldots, \chi_m, \xi_1, \ldots, \xi_n$ are all atoms (i.e. propositional variables or the...
negates thereof), and with some atom $\chi_i$ being the same as some atom $\xi_j$. Belnap remarks that tautological entailmenthood is effectively decidable, and observes strong equality between the set of first-degree theorems of the system $E$ (of entailment) and the set of tautological entailments, see also [5].

In [2, Section 15.2] this set was formalized by a proof system operating with the first-degree entailments as primitive expressions. This system was called there $E_{fde}$ to emphasize that it presents, in fact, the first-degree entailment fragment of the calculus $E$. Dunn in [12] uses the label $R_{fde}$, because the first-degree entailment fragments of systems $R$ and $E$ are the same. Taking into account that, on an object language level, the semantic relation of entailment is often represented by the consequence sign ($\vdash$), we reproduce here this formalism as a “binary consequence system”, the expressions of which are all of the form $\varphi \vdash \psi$, to be read as “$\varphi$ has $\psi$ as a consequence” (see, e.g., [13, p. 302]). We will refer to this system as $FDE$, which is most common nowadays. It consists of initial consequences taken as axioms, and also rules for transforming one consequences into the others:

System $FDE$:  

\begin{itemize}
  \item $a_1_{fde}$. $\varphi \land \psi \vdash \varphi$
  \item $a_2_{fde}$. $\varphi \land \psi \vdash \psi$
  \item $a_3_{fde}$. $\varphi \vdash \varphi \lor \psi$
  \item $a_4_{fde}$. $\varphi \lor \psi \vdash \varphi \lor \psi$
  \item $a_5_{fde}$. $\varphi \land (\psi \lor \chi) \vdash (\varphi \land \psi) \lor \chi$
  \item $a_6_{fde}$. $\varphi \vdash \neg \neg \varphi$
  \item $a_7_{fde}$. $\neg \neg \varphi \vdash \varphi$
  \item $r_1_{fde}$. $\varphi \vdash \psi$; $\psi \vdash \chi / \varphi \vdash \chi$
  \item $r_2_{fde}$. $\varphi \vdash \psi$; $\varphi \vdash \chi / \varphi \vdash \psi \land \chi$
  \item $r_3_{fde}$. $\varphi \vdash \chi$; $\psi \vdash \chi / \varphi \lor \psi \vdash \chi$
  \item $r_4_{fde}$. $\varphi \vdash \psi / \sim \psi \vdash \sim \varphi$.
\end{itemize}

Note, that four De Morgan laws ($\neg (\varphi \land \sim \psi) \vdash \sim (\varphi \lor \psi)$, $\sim (\varphi \lor \psi) \vdash \sim (\varphi \land \sim \psi)$, $\sim (\varphi \lor \psi) \vdash \sim (\varphi \land \sim \psi)$) are derivable in $FDE$. Alternatively, these laws can be taken as initial postulates, whereby $r_4_{fde}$ can be excluded from the list of initial rules, remaining admissible (see [14, pp. 14–15]).

Dunn in [10] initiated a highly innovative research program for semantic justification of the first-degree entailments, culminating in his paper [11]. The main point of the program consists in allowing underdetermined and overdetermined valuations that can in certain situations falsify logical laws