WEIGHT FUNCTION FOR THE QUANTUM AFFINE ALGEBRA $U_q(\widehat{\mathfrak{sl}_3})$

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We give a precise expression for the universal weight function of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_3})$. The calculations use the technique of projecting products of Drinfeld currents on the intersections of Borel subalgebras.

Keywords: quantum affine algebra, current realization, nested Bethe ansatz, weight function, biorthogonal decompositions of Borel current subalgebras, projections on intersections of different Borel subalgebras

1. Introduction

The ideology of a nested Bethe ansatz [1] prescribes two steps for describing the transfer-matrix eigenvectors in finite-dimensional representations of a quantum affine algebra. First, specific rational functions with values in the representation should be constructed; second, a system of Bethe equations for these functions should be solved. These rational vector-valued functions are called off-shell (nested) Bethe vectors. They can serve as a generating system of vectors of a finite-dimensional representation of a quantum affine algebra. We use the equivalent name “weight function,” which came from applications in difference Knizhnik–Zamolodchikov equations [2].

A general construction of a weight function for a quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ was recently suggested [3]. This construction uses the existence of two different types of Borel subalgebras in a quantum affine algebra. One type is related to the realization of $U_q(\widehat{\mathfrak{g}})$ as a quantized Kac–Moody algebra, and the other comes from the current realization of $U_q(\widehat{\mathfrak{g}})$ proposed by Drinfeld [4]. The weight function is defined as the projection of a product of Drinfeld currents on the intersection of Borel subalgebras of $U_q(\widehat{\mathfrak{g}})$ of different types (see Sec. 3.1).

Our goal in this paper is to develop a technique for calculating the weight function starting from the definition in [3]. According to this definition, for calculating the weight function, the product of Drinfeld currents must be arranged in a normally ordered form. Then only those terms are kept that belong to the intersection of Borel subalgebras of different types. The normal-ordering procedure requires investigating the current adjoint action and the composed root currents, introduced in [5]. The final result is a precise universal expression for the weight function of $U_q(\widehat{\mathfrak{sl}_3})$, which can then be specialized to any finite-dimensional representation of $U_q(\widehat{\mathfrak{sl}_3})$. We note that the calculation of the weight function for $U_q(\widehat{\mathfrak{gl}_N})$ at the level of a tensor product of evaluation representations can be found in [6].

This paper is organized as follows. In Sec. 2, we introduce the main objects of the investigation. Section 3 is devoted to formulating the main results. They contain a precise expression for the weight

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function of $U_q(\hat{\mathfrak{sl}}_3)$ (Theorems 3.2 and 3.3). As a particular case, we give an expression for the weight function of $U_q(\hat{\mathfrak{sl}}_2)$ in an integral form (Theorem 3.3). The kernel of the integral is a well-known partition function, which coincides with the partition function of the six-vertex model on a finite square lattice with fixed domain-wall boundary conditions. Later, we need a combinatorial identity for this kernel, which we prove by observing the self-adjointness of the projection operators (Proposition 3.4).

Sections 4 and 5 are devoted to proving the main statements, which includes studying the analytic properties of composed currents and related products of currents (strings) and of current adjoint actions. We also note an important role of symmetrization procedures, based on the properties of the analytic continuation of the products of currents and of their projections (see Proposition 5.1). In the appendices, we give the necessary properties of the opposite projection operator, commutation relations between currents and their projections, and another proof of the main result.

2. Basic notation

2.1. $U_q(\hat{\mathfrak{sl}}_3)$ in Chevalley generators. The quantum affine algebra $U_q(\hat{\mathfrak{sl}}_3)$ is generated by Chevalley generators $e_{\pm \alpha_i}$ and $k_{\pm1}^{\alpha_i}$, where $i = 0, 1, 2$ and $\prod_{i=0}^2 k_{\alpha_i} = 1$, subject to the relations

$$k_{\alpha_i} e_{\pm \alpha_j} k_{\alpha_i}^{-1} = q^{\pm \alpha_i} e_{\pm \alpha_j}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{k_{\alpha_i} - k_{-1}^{-\alpha_i}}{q - q^{-1}}, \quad (2.1)$$

$$e_{\pm \alpha_j} e_{\pm \alpha_j} + 2 [q e_{\pm \alpha_i}, e_{\pm \alpha_i}, e_{\pm \alpha_j}] = 0, \quad i \neq j, \quad (\alpha_i, \alpha_j) = -1, \quad (2.2)$$

where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ is the Gauss $q$-number and $a_{ij} = (\alpha_i, \alpha_j)$ is a symmetrized Cartan matrix of the affine algebra $\hat{\mathfrak{sl}}_3$, 

$$a_{ij} = (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (2.3)$$

One of the possible Hopf structures (which we call the standard Hopf structure) is given by the formulas

$$\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i}, \quad \Delta(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{-1}^{-\alpha_i},$$

$$\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i}, \quad \epsilon(e_{\pm \alpha_i}) = 0, \quad \epsilon(k_{\alpha_i}^{\pm1}) = 1, \quad (2.4)$$

$$a(e_{\alpha_i}) = -k_{-1}^{\alpha_i} e_{\alpha_i}, \quad a(e_{-\alpha_i}) = -e_{-\alpha_i} k_{\alpha_i}, \quad a(k_{\alpha_i}^{\pm1}) = k_{\alpha_i}^{\mp1},$$

where $\Delta$, $\epsilon$, and $a$ are the respective comultiplication, counit, and antipode maps.

2.2. Current realization of the algebra $U_q(\hat{\mathfrak{sl}}_3)$. As does any quantum affine algebra, $U_q(\hat{\mathfrak{sl}}_3)$ admits a current realization [4]. In this description (we again assume that the central charge is zero), $U_q(\hat{\mathfrak{sl}}_3)$ is generated by the elements $e_i[n]$ and $f_i[n]$, where $i = \alpha, \beta$ and $n \in \mathbb{Z}$, and $k_{i}^{\pm1}$ and $h_i[n]$, where $i = \alpha, \beta$ and $n \in \mathbb{Z} \setminus \{0\}$. They are gathered in the generating functions

$$e_i(z) = \sum_{n \in \mathbb{Z}} e_i[n] z^{-n}, \quad f_i(z) = \sum_{n \in \mathbb{Z}} f_i[n] z^{-n},$$

$$\psi_i^{\pm}(z) = \sum_{n \geq 0} \psi_i^{\pm}[n] z^{\mp n} = k_i^{\pm1} \exp \left( \pm (q - q^{-1}) \sum_{n > 0} h_i[\pm n] z^{\mp n} \right), \quad (2.5)$$

1In what follows, we do not use a grading operator and set the central charge equal to zero. Such an algebra is usually denoted by $U'_q(\mathfrak{sl}_3)$.  

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