THE PHASE DIAGRAM OF A DIRECTED POLYMER IN RANDOM MEDIA WITH $p$-SPIN FERROMAGNETIC INTERACTIONS

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We consider a directed polymer model with an additive $p$-spin ($p \geq 2$) ferromagnetic term in the Hamiltonian. We give a rigorous proof for the specific free energy and derive the phase diagram. This model was proposed previously, and a detailed proof was given in the case $p = 2$, while the main result was only stated for $p > 2$. We give a detailed proof of the main result and show the behavior of the model as $p \to \infty$ by constructing the phase diagram also in this case. These results are important in many applications, for instance, in telecommunication and immunology. Our major finding is that in the phase diagram for $p > 2$, a new transition curve (absent for $p = 2$) emerges between the paramagnetic region and the so-called mixed region and that the ferromagnetic region diminishes as $p \to \infty$.

Keywords: directed polymer, large deviation, phase diagram

1. Introduction

Directed polymers in random media (self-avoiding random walks in random media or random trees [1]) are widely applied not only in the physical sciences but also in many other fields (e.g., social opinion dynamics and financial markets [2]–[5], evolutionary dynamics [6], [7], diffusion limited aggregation [8], shocks in one-dimensional turbulence [9], [10], immunology [11]–[13], etc.). A directed polymer in a disordered tree is an example of diffusion in disordered media [14]–[16] and is Derrida’s random energy model (REM) [17]–[21], which consists of $z^n$ independent identically distributed random energy levels for a tree dividing into $z$ branches at each node (see Fig. 1 for the case $z = 2$).

Indeed, it was shown in [22], [19] that the REM is the limit case of the Sherrington–Kirkpatrick model with interaction of $r$ spins as $r \to \infty$. The $r$-spin interaction is defined by the Hamiltonian

$$-\mathcal{H}_{\text{REM}} = \sum_{i_1 < i_2 < \cdots < i_r = 1}^n J_{i_1, i_2, \ldots, i_r} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_r},$$

where $J_{i_1, i_2, \ldots, i_r}$ are independent identically distributed Gaussian random variables (quenched, which describes the exchange integrals) and $\sigma_{i_1}, \ldots, \sigma_{i_r}$ are Ising spins. If $r$ is small ($r \ll n$), then the energies due to $r$-group interactions can be expected to be strongly correlated, while these correlations become negligible as $r \to \infty$, in which case we recover the REM with the resulting $2^n$ independent identically distributed energy levels, in which case

$$-\mathcal{H}_{\text{REM}} = \sum_{i=1}^n V_i.$$ (1)

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(a) An example situation yielding $2^n$ possible energy levels: the Ising spins $\sigma_1, \ldots, \sigma_n$ ($n = 5$ here) have a ferromagnetic effect on the system from a $p$-spin group ($p = 3$) determined by $\sum_{i=1}^{5} h_i \sum_{i<j<k}^{5} \sigma_i \sigma_j \sigma_k + (\sum_{i=1}^{5} \sigma_i)^3 - (\text{lower degree terms})$. (b) The tree structure of energy levels: $k_{j+1} = 2k_j - (1 - \sigma_{j+1})/2$, $k_j = 1, 2, \ldots, n$, and $\sigma_i = \pm 1$ in the case $n = 3$; the total system energy is the sum of the energy due to each spin configuration and the energy due to the interaction between $p$-groups (only the first three levels of this path are shown).

Our main object of interest is a directed polymer, which is defined as a self-avoiding random walk along a tree that branches into $z=2$ random directions at each node and has the base Hamiltonian the same as REM Hamiltonian (1).

We consider the example shown in Fig. 1, where we find $2^n$ independent identically distributed energy levels, each corresponding to one of the $2^n$ walks (or paths by our definition; see below) ending after $n$ steps along the tree, shown in Fig. 1b. Such a walk can be obtained, for example, by considering a system of $n$ localized Ising spins $\sigma_1, \sigma_2, \ldots, \sigma_n$. We can define the energy of each walk as the sum of the energies corresponding to each branch the walk traverses. We can then define a Hamiltonian as

$$-\mathcal{H}_1 := \sum_{j=1}^{n} V_{j,(\sigma)^j}$$  \hspace{1cm} (2)$$

for a given configuration of spins, for example, $(\sigma)^j$, and for $V$ having some prior probability distribution depending on a single parameter, for example, $\gamma$. In addition, each $p$-group of these spins exerts a ferromagnetic effect (due to some external field or to the interaction between $p$-groups; see Fig. 1), which we describe by the Hamiltonian

$$-\mathcal{H}_2 := \frac{1}{n^p-1} \sum_{k=1}^{M} h_k \left| \sum_{j=1}^{n} \sigma_j^{(k)} \right|^p, \quad p > 1,$$  \hspace{1cm} (3)$$

which can be simplified to

$$-\mathcal{H}_2 = \frac{1}{n^p-1}(h_1 + h_2 + \cdots + h_M) \left| \sum_{j=1}^{n} \sigma_j \right|^p = \frac{1}{n^p-1} \lambda \left| \sum_{j=1}^{n} \sigma_j \right|^p$$