RENORMALIZATION IN THE CAUCHY PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION

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We consider the Cauchy problem for the Korteweg–de Vries equation with a small parameter at the highest derivative and a large gradient of the initial function. We construct an asymptotic solution of this problem by the renormalization method.

Keywords: Korteweg–de Vries equation, Cauchy problem, asymptotic solution, renormalization

1. Introduction

We consider the Cauchy problem for the Korteweg–de Vries equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \varepsilon \frac{\partial^3 u}{\partial x^3} = 0, \quad t \geq 0, \quad \varepsilon > 0, \tag{1}
\]

\[
u(x, 0, \varepsilon, \rho) = \Lambda \left( \frac{x}{\rho} \right), \quad x \in \mathbb{R}, \quad \rho > 0, \tag{2}
\]

with a bounded initial function \( \Lambda \) that has finite limits

\[
\Lambda^\pm_0 = \lim_{s \to \pm \infty} \Lambda(s), \quad \Lambda^+_0 < \Lambda^-_0,
\]

and a derivative that vanishes sufficiently rapidly at infinity. This is a classical model of nonlinear wave propagation in a medium with small dispersion. In the case of a discontinuous initial function, an asymptotic solution was obtained in [1]. For steplike initial data, asymptotic formulas were found by the inverse spectral transform method [2]. Under certain restrictions on the initial function, the asymptotic behavior can be studied by the Whitham method, as in [3], [4], where the relation to the inverse spectral transform method was also demonstrated.

Here, we construct the solution of problem (1), (2) asymptotic in the parameters \( \varepsilon \) and \( \rho \) for all \( x \) and finite \( t \). It is clear that the structure of the asymptotic solution must depend essentially on the balance of the parameters \( \varepsilon \) and \( \rho \). We assume the condition

\[
\mu = \frac{\rho}{\sqrt{\varepsilon}} \to 0.
\]

It is known that the behavior of solutions of singularly perturbed differential equations with a small parameter at the highest derivative in certain cases becomes self-similar in a sense. Analysis by the renormalization method is a tool for investigating such problems. Here, the condition of smallness of the parameter \( \varepsilon \) at the highest derivative is taken into account. In this paper, we consider an asymptotic solution of the Cauchy problem for the Korteweg–de Vries equation with a small parameter in the highest derivative and a large gradient of the initial function. The renormalization method is used to construct an asymptotic solution of this problem.

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ization method [5] then becomes effective. The advantage of this approach is that a uniform approximation of the problem can be obtained by immediately removing the requirement to construct the asymptotic ansatz in separate regions. For instance, a composite asymptotic solution of the Cauchy problem with condition (2) for the quasilinear parabolic equation was obtained in [6] by the matching method [7], and it was shown in [8] that the renormalization approximation of the solution is asymptotically close to the composite asymptotic solution.

2. Asymptotic solution

We construct an asymptotic solution of problem (1), (2) using a technique similar to the renormalization group method in its simplest variant. We pass to the inner variables

\[ x = \sqrt{\varepsilon} \eta, \quad t = \sqrt{\varepsilon} \theta, \]

because this allows taking all terms in Eq. (1) into account. As a “bare” function, we take the solution of the equation

\[ \frac{\partial Z}{\partial \theta} + Z \frac{\partial Z}{\partial \eta} + \frac{\partial^3 Z}{\partial \eta^3} = 0 \] (4)

with the initial condition

\[ Z(\eta, 0) = \begin{cases} \Lambda_0^-, & \eta < 0, \\ \Lambda_0^+, & \eta > 0. \end{cases} \] (5)

We seek the expansion of the solution in the form

\[ u(x, t, \varepsilon, \rho) = Z(\eta, \theta) + \mu W(\eta, \theta, \mu) + \ldots, \] (6)

where the correction \( \mu W(\eta, \theta, \mu) \) must remove the singularity of the bare function at the initial instant. It follows from Eqs. (1) and (4) that the function \( W \) satisfies the linear equation

\[ \frac{\partial W}{\partial \theta} + \frac{\partial (ZW)}{\partial \eta} + \frac{\partial^3 W}{\partial \eta^3} = 0. \] (7)

Differentiating (4) with respect to \( \eta \), we verify that the function

\[ G(\eta, \theta) = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \frac{\partial Z(\eta, \theta)}{\partial \eta} \]

satisfies (7). Moreover, \( G \) represents the Green’s function because

\[ \lim_{\theta \to +0} \int_{-\infty}^{\infty} G(\eta, \theta) f(\eta) \, d\eta = -\frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} Z(\eta, 0) f'(\eta) \, d\eta = f(0) \]

for any function \( f \) of compact support.

We select the solution \( W \) in the form of the convolution product with the Green’s function \( G \) such that the asymptotic solution satisfies initial condition (2). Then

\[ W = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} \frac{\partial Z(\eta - \mu s, \theta)}{\partial \eta} [\Lambda(s) - Z(s, 0)] \, ds. \]