REAL PROJECTIVE CONNECTIONS, V. I. SMIRNOV’S APPROACH, AND BLACK-HOLE-TYPE SOLUTIONS OF THE LIOUVILLE EQUATION

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We consider real projective connections on Riemann surfaces and their corresponding solutions of the Liouville equation. We show that these solutions have singularities of a special type (a black-hole type) on a finite number of simple analytic contours. We analyze the case of the Riemann sphere with four real punctures, considered in V. I. Smirnov’s thesis (Petrograd, 1918) in detail.

Keywords: uniformization, Riemann surface, projective connection, Fuchsian projective connection, monodromy group, Liouville equation, Liouville action, singular solution

Dedicated to my teacher Ludvig Dmitrievich Faddeev on the occasion of his 80th birthday

1. Introduction

One of the central problems of mathematics in the second half of the 19th century and at the beginning of the 20th century was the problem of uniformization of Riemann surfaces. The classics, Klein [1] and Poincaré [2], associated it with studying second-order ordinary differential equations with regular singular points. Poincaré proposed another approach to the uniformization problem [3]. It consists in finding a complete conformal metric of constant negative curvature, and it reduces to the global solvability of the Liouville equation, a special nonlinear partial differential equation of elliptic type on a Riemann surface.

Here, we illustrate the relation between these two approaches and describe solutions of the Liouville equation corresponding to second-order ordinary differential equations with a real monodromy group. In the modern physics literature on the Liouville equation, it is rather commonly assumed that for the Fuchsian uniformization of a Riemann surface, it suffices to have a second-order ordinary differential equation with a real monodromy group. But the classics already knew that this is not the case, and they analyzed second-order ordinary differential equations with a real monodromy group on genus-0 Riemann surfaces with punctures in detail. Nonetheless, they did not consider the relation to the Liouville equation, and we partially fill this gap here.

Namely, in Sec. 2, following the lectures [4], we briefly describe the theory of projective connections on a Riemann surface—an invariant method for defining a corresponding second-order ordinary differential equation with regular singular points. Following [5], [6], we review the main results on the Fuchsian uniformization, the Liouville equation, and the complex geometry of the moduli space. In Sec. 3, following [7], we present the modern classification of projective connections with a real monodromy group and review the results of V. I. Smirnov’s thesis [8] (Petrograd, 1918). This work, published in [9], [10], was the first
where a complete classification of equations with a real monodromy group was given in the case of four real punctures. In Sec. 3.2, we give a modern interpretation of Smirnov’s results. Finally, in Sec. 4, we describe solutions of the Liouville equation with black-hole-type singularities associated with real projective connections. To the best of our knowledge, these solutions have not been considered previously.

2. Projective connections, uniformization, and the Liouville equation

2.1. Projective connections. Let $X_0$ be a compact genus-$g$ Riemann surface with marked points $x_1, \ldots, x_n$, where $2g + n - 2 > 0$, and let $\{U_\alpha, z_\alpha\}$ be a complex-analytic atlas with local coordinates $z_\alpha$ and transition functions $z_\alpha = g_{\alpha\beta}(z_\beta)$ on $U_\alpha \cap U_\beta$. Let $X = X_0 \setminus \{x_1, \ldots, x_n\}$ denote a corresponding Riemann surface of type $(g, n)$, a genus-$g$ surface with $n$ punctures. The collection $R = \{r_\alpha\}$, where $r_\alpha$ are holomorphic functions on $U_\alpha \cap X$, is called a (holomorphic) projective connection on $X$ if on every intersection $U_\alpha \cap U_\beta \cap X$,

$$r_\beta = r_\alpha \circ g_{\alpha\beta}(g_{\alpha\beta}')^2 + S(g_{\alpha\beta}),$$

where $S(f)$ is the Schwarzian derivative of a holomorphic function $f$,

$$S(f) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$

In addition, we assume that if $x_i \in U_\alpha$ and $z_\alpha(x_i) = 0$, then

$$r_\alpha(z_\alpha) = \frac{1}{2z_\alpha^2} + O\left( \frac{1}{|z_\alpha|} \right), \quad z_\alpha \to 0. \quad (1)$$

Projective connections form an affine space $P(X)$ over the vector space $Q(X)$ of holomorphic quadratic differentials on $X$; elements of $Q(X)$ are collections $Q = \{q_\alpha\}$ with the transformation law

$$q_\beta = q_\alpha \circ g_{\alpha\beta}(g_{\alpha\beta}')^2$$

and the additional condition that $q_\alpha(z_\alpha) = O(|z_\alpha|^{-1})$ as $z_\alpha \to 0$ if $x_i \in U_\alpha$ and $z_\alpha(x_i) = 0$. The vector space $Q(X)$ has the complex dimension $3g - 3 + n$ (for more details on projective connections and quadratic differentials, see [4] and the references therein).

A projective connection $R$ naturally determines a second-order linear differential equation on the Riemann surface $X$, the Fuchsian differential equation

$$\frac{d^2 u_\alpha}{dz_\alpha^2} + \frac{1}{2}r_\alpha u_\alpha = 0, \quad (2)$$

where $U = \{u_\alpha\}$ is understood as a multivalued differential of order $-1/2$ on $X$. Equation (2) determines the monodromy group, a representation of the fundamental group $\pi_1(X, x_0)$ of the Riemann surface $X$ with the marked point $x_0$ in $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{ \pm I \}$. Condition (1) implies that the standard generators of $\pi_1(X, x_0)$, which correspond to the loops around the punctures $x_i$, are mapped to parabolic elements in $\text{PSL}(2, \mathbb{C})$ under the monodromy representation.

2.2. Uniformization. According to the uniformization theorem

$$X \cong \Gamma \setminus \mathbb{H}, \quad (3)$$

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