GENERALIZED PASCAL’S TRIANGLES AND SINGULAR ELEMENTS OF MODULES OF LIE ALGEBRAS

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We consider the problem of determining the multiplicity function \( m^{\otimes_p\omega}_\xi \) in the tensor power decomposition of a module of a semisimple algebra \( g \) into irreducible submodules. For this, we propose to pass to the corresponding decomposition of a singular element \( \Psi((L^\omega_g)^{\otimes_p}) \) of the module tensor power into singular elements of irreducible submodules and formulate the problem of determining the function \( M^{\otimes_p\omega}_\xi \). This function satisfies a system of recurrence relations that corresponds to the procedure for multiplying modules. To solve this problem, we introduce a special combinatorial object, a generalized \((g,\omega)\) pyramid, i.e., a set of numbers \( (p,\{m_i\})_{g,\omega} \) satisfying the same system of recurrence relations. We prove that \( M^{\otimes_p\omega}_\xi \) can be represented as a linear combination of the corresponding \((p,\{m_i\})_{g,\omega} \). We illustrate the obtained solution with several examples of modules of the algebras \( s(3) \) and \( s(5) \).

Keywords: theory of Lie algebra representation, tensor product of modules, Weyl formula

1. Graph of a singular element of a module and a generalized \((g,\omega)\) pyramid

1.1. Statement of the problem. Let \( g \) be a semisimple Lie algebra of rank \( n \). We let \( L^\omega_g \) denote the irreducible submodule of \( g \) with a highest weight \( \omega \). We study the tensor powers \((L^\omega_g)^{\otimes_p}_{p \in \mathbb{Z}_+} \) and their decomposition into irreducible submodules

\[
(L^\omega_g)^{\otimes_p} = \sum_\xi m^{\otimes_p\omega}_\xi L^\xi_g. \tag{1}
\]

The problem is to determine the multiplicities \( m^{\otimes_p\omega}_\xi \) as functions of \( \xi \) and \( p \).

We regard the function \( m^{\otimes_p\omega}_\xi \) as the multiplicity of the highest weight of the module \( L^\xi_g \) in the weight diagram \( N((L^\omega_g)^{\otimes_p}) \). The function \( m^{\otimes_p\omega}_\xi \) is defined on the dominant-weight lattice \( P^+ \). The dominant-weight lattice is in the closure of the fundamental Weyl chamber \( C(0) \). We construct the natural continuation of the function \( m^{\otimes_p\omega}_\xi \) to other Weyl chambers. If this continuation is regarded as the Weyl anti-invariant function

\[
M^{\otimes_p\omega}_{w(\xi+\rho)-\rho}\big|_{w \in W} = \det(w)m^{\otimes_p\omega}_\xi, \tag{2}
\]

then the function \( M^{\otimes_p\omega}_\xi \) describes the multiplicities of weights of a singular element of the module \( L^\xi_g \), and the original problem can be easily rewritten in terms of singular elements as

\[
\Psi((L^\omega_g)^{\otimes_p}) = \sum_\xi M^{\otimes_p\omega}_\xi \Psi(L^\xi_g).
\]

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The domain of the function $M^p_\omega$ is the entire weight lattice $P$. The required function $m^p_\omega$, which is a solution of problem (1), can be obtained using the restriction $M^p_\omega|_{C(0)}$.

The problem of determining $M^p_\omega$ is considered below.

It was shown in [1], [2] that the functions $M^p_\omega$ with different $p$ are connected by the system of recurrence relations

$$
\sum_{\xi \in P} M^p_\omega e^\xi = N(L^\omega) \sum_{\gamma \in P} M^p_\nu e^\gamma,
$$

where $N(L^\omega)$ is the module diagram in the algebra of formal exponentials. We can regard this as a recurrence relation for the singular elements because the right-hand expression is $N(L^\omega)\Psi((L^\omega)^{\otimes p-1})$.

**Definition 1.** The graph $S^\omega_p$ of a singular element is an infinite graph $S^\omega_p = (Q, T)$ defined by the vertex set $Q = \bigcup_{p \geq 0} Q_p$ and the edge set $T = \bigcup_{p \geq 1} T_p$, which form a disjunctive union of subsets $Q_p$ and $T_p$ with the following properties:

1. We have $Q_0 = \{ v(0, \xi) : \xi \in P, \xi = w(\rho) - \rho \}$, where $w$ is an element of the Weyl group of the algebra $g$.

2. The subsets $Q_p$ and $T_p$ are finite,

$$
Q_p = \{ v(p, \xi) : \xi \in P, \xi \in \Psi((L^\omega)^{\otimes p}) \},
$$

and $T_p$ is given by a recurrence relation and is the set of edges connecting the vertices $v(p, \xi) \in Q_p$ with the vertices $\hat{v}(p + 1, \xi + \nu) \in Q_{p+1}$, where $\nu \in N(L^\omega)$. A pair $(Q_p, T_p)$ (and also the set $Q_p$) is called the level $p$ of the graph $S^\omega_p$.

3. A return map $r$ and a continue map $s$ from $R$ into $Q$ are given such that $r(T_p) = Q_p$, $s(T_p) = Q_{p-1}$ and $s^{-1}(v) \neq \emptyset$, $r^{-1}(\hat{v}) \neq \emptyset$ for any $v \in Q$ and $\hat{v} \in Q \setminus Q_0$. These maps are also given by a recurrence relation, i.e., each edge $t(p+1) \in T_{p+1}$ connects the vertex $v(p, \xi) \in Q_p$ to the vertex $\hat{v}(p+1, \xi) \in Q_{p+1}$, and hence $r(t(p+1)) = \hat{v}$ and $s(t(p+1)) = v$.

At the vertices $v(p, \xi)$ of the graph $S^\omega_p$, we introduce a function $M^p_\omega$ to associate each vertex $v(p, \xi)$ of the graph with the singular multiplicity of the corresponding weight $\xi$ in $\Psi((L^\omega)^{\otimes p})$.

**Definition 2.** A generalized $(g, \omega)$ pyramid is a set of numbers $\{p, \{m_i\}\}_{g, \omega}$, where $p = -\infty, \ldots, \infty$, $m_i = 0, \ldots, \infty$, and $i = 0, \ldots, n$, whose elements satisfy the recurrence relation

$$
\sum_{\xi \in P} (p, \{m_i\})_{g, \omega} e^\xi = N(L^\omega) \sum_{\gamma \in P} (p - 1, \{l_i\})_{g, \omega} e^\gamma
$$

with the boundary conditions $(0, \{0\}) = 1$.

Here, $\{m_i\}$ and $\{l_i\}$ are the coordinates of the vectors $\xi$ and $\gamma$ in the weight lattice basis, which is said to be combinatorial. A combinatorial basis is a set of vectors drawn from the highest weight of the module under study along the boundaries of its diagram. The lengths of the vectors in the combinatorial basis are chosen such that all weights of the module have integer nonnegative coordinates in this basis. In expression (4), the numbers $\{m_i\}$ are the combinatorial coordinates of the weight $\xi$ in the basis of the module $(L^\omega)^{\otimes p}$, and $\{l_i\}$ are the combinatorial coordinates of the weight $\gamma$ in the basis of the module $(L^\omega)^{\otimes p-1}$. 

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