We consider analogs of the Galerkin method for a linear wave equation of the fifth order with generalized functions on the right-hand side. Theorems on the convergence of an approximate method, depending on the order of singularity of the right-hand side, are proved.

We study a linear partial differential equation with fifth-order derivatives

\[ \mathcal{L} u \equiv A(u_{tt}) + B^2(u_t) + DC(u) = f, \]

where \( A, B, C, \) and \( D \) are second-order differential operators with respect to space variables.

Equations of the type (1) arise, e.g., in the course of the investigation of the dynamics of plane motions of an incompressible viscous liquid [1–3]:

\[ \frac{\partial^2}{\partial t^2} \Delta u - v \frac{\partial}{\partial t} \Delta^2 u + \omega_0 \frac{\partial^2}{\partial \xi_1^2} u = f, \quad \Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}. \]

Equation (1) also describes linear waves on a screw flow [4], small-amplitude oscillations of a mobile stratified liquid [5], and many other wave processes.

Equation (1) generalizes equations of the Sobolev type \((B = 0, D = 1)\). Performing simple changes of variables, one can easily indicate individual cases where Eq. (1) reduces to pseudoparabolic, or pseudohyperbolic, or classical parabolic and hyperbolic equations.

The first studies of wave processes of this type related mainly to stationary modes or were restricted to the investigation of domains of special form [5–7]. Fairly general investigations of boundary-value problems for Eq. (1) were carried out in [8–11]. For certain operators \( A, B, C, \) and \( D \), the unique solvability of the problem with regular initial data was proved in [10, 11] and its generalized solvability in the class of generalized functions of finite order was proved in [8, 9]. In the case where the right-hand side is a generalized function of some finite order, for certain types of operators (1) the generalized solvability was proved and some optimization problems were studied in [8, 12, 13]. Note that these investigations were carried out for an operator \( \mathcal{L} \) acting in a pair of Sobolev spaces in the case of nonnegative operators \( A, B, C, \) and \( D \) without lower-order terms.

In the present work, we propose an approximate method of the Galerkin type for the solution of the wave equation (1) and study the convergence of this method. The study is based on the proof of chains of \textit{a priori} estimates for some extensions of the operator \( \mathcal{L} \). In the proof of lower bounds, we use auxiliary integro-differential operators that are new for such problems. This allows us to establish a countable scale (with respect to the smoothness of the right-hand side of the equation) of theorems on the convergence of the Galerkin method for

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operators \( A, B, C, \) and \( D \) with lower-order terms and to improve the smoothness of a solution of the problem. The necessity in results of this type arises, e.g., in the course of the solution of problems of optimal control of systems with concentrated (in particular, pulse) influences [13].

1. Main Notation and Spaces

In a cylindrical domain \( (t, \xi) \in Q = (0, T) \times \Omega, \) where \( \Omega \subset \mathbb{R}^n \) is a bounded simply-connected domain with regular boundary \( \partial \Omega \), we consider the general wave equation (1). Assume that \( u(t, \xi) \) is a function of state that satisfies the boundary conditions

\[
\begin{align*}
    u \big|_{t=0} &= u_t \big|_{v=0} = 0, \\
    u \xi_{\partial} \in \Omega &= \frac{\partial u}{\partial \nu_B} \big|_{\xi_{\partial}} = 0, \\
\end{align*}
\]

\( \bar{\mu}_B = B \bar{n} \) is the vector of the conormal to the surface \( \partial \Omega \), \( B = \{b_{ij}(\xi)\}_{i,j=1}^n \) is the matrix of coefficients of the operator \( B \), and \( \bar{n} \) is the vector of the exterior normal to the surface \( \partial \Omega \).

The operator \( A \) does not depend on the variable \( t \) and is defined by the second-order differential expression

\[
A(u) \equiv - \sum_{i,j=1}^n \frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi) \frac{\partial u}{\partial \xi_j} \right) + \sum_{i=1}^n a_i(\xi) \frac{\partial u}{\partial \xi_i} + a(\xi) u;
\]

the operators \( B, C, \) and \( D \) are defined by analogy.

Let \( L_0 \) be the set of functions infinitely differentiable in \( Q \) and satisfying the conditions

\[
\begin{align*}
    u \big|_{t=0} &= u_t \big|_{v=0} = \ldots = 0, \\
    u \xi_{\partial} \in \Omega &= \frac{\partial u}{\partial \nu_B} \big|_{\xi_{\partial}} = 0,
\end{align*}
\]

and let \( W_0^1, H_0^1, \) and \( V_0^1 \) be the completions of \( L_0 \) with respect to the norms

\[
\|u\|_{W_0^2}^2 = \sum_{i,j=1}^n \int_Q u_{\xi_i \xi_j}^2 \, dQ,
\]

\[
\|u\|_{H_0^2}^2 = \sum_{i=1}^n \int_Q u_{\xi_i}^2 \, dQ + \sum_{i,j=1}^n \int_Q u_{\xi_i \xi_j}^2 \, dQ,
\]

\[
\|u\|_{V_0^2}^2 = \|u\|_{H_0^2}^2 + \sum_{i,j=1}^n \int_{\Omega} u_{\xi_i \xi_j}^2 \big|_{t=T} \, d\Omega.
\]

Analogously, \( L_T \) is the set of functions infinitely differentiable in \( Q \) and satisfying the conjugate conditions

\[
\begin{align*}
    v \big|_{t=T} &= v_t \big|_{v=T} = \ldots = 0, \\
    v \xi_{\partial} \in \Omega &= \frac{\partial v}{\partial \nu_B} \big|_{\xi_{\partial}} = 0,
\end{align*}
\]