ON MODULI OF SMOOTHNESS AND FOURIER MULTIPLIERS IN \( L_p, 0 < p < 1 \)

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We prove a theorem on the relationship between the modulus of smoothness and the best approximation in \( L_p, 0 < p < 1 \), and theorems on the extension of functions with preservation of the modulus of smoothness in \( L_p, 0 < p < 1 \). We also give a complete description of multipliers of periodic functions in the spaces \( L_p, 0 < p < 1 \).

Let \( A \) be either the real axis \( \mathbb{R} \), or the unit circle \( \mathbb{T} \) realized as the segment \([0, 2\pi]\) with identified endpoints, or a finite segment. Let \( \mathcal{T}_n = L_p(A) \), \( 0 < p \leq \infty \), denote the space of measurable functions \( f \) such that \( \|f\|_p = \|f\|_p(A) < \infty \), where

\[
\|f\|_p(A) := \left\{ \begin{array}{ll}
\left( \frac{1}{p} \int_A |f(x)|^p \, dx \right)^{\frac{1}{p}} & \text{if } 0 < p < \infty, \\
\vrai \sup_{x \in A}|f(x)| & \text{if } p = \infty.
\end{array} \right.
\]

Let \( \mathcal{T}_n \) be the set of trigonometric polynomials of degree not higher than \( n \). For \( 2\pi \)-periodic functions \( f \), we denote by

\[
E_n(f)_p := \inf_{T \in \mathcal{T}_n} \|f - T\|_p(\mathbb{T})
\]

the best approximation of the function \( f \) by the set \( \mathcal{T}_n \).

The modulus of smoothness of a function \( f \in L_p(A) \) of order \( r \) with step \( h \) is defined by the relation

\[
\omega_r(f, h)_p := \omega_r(f, h)_p(A) := \sup_{0 < \delta \leq h} \left( \int_A |\Delta^r_{\delta} f(x)|^p \, dx \right)^{\frac{1}{p}},
\]

where

\[
\Delta^r_{\delta} f(x) := \begin{cases}
\sum_{s=0}^{r} (-1)^s \binom{r}{s} f(x + s\delta) & \text{if } [x, x + r\delta] \subset A, \\
0 & \text{if } [x, x + r\delta] \not\subset A.
\end{cases}
\]
An upper bound for $E_n(f)_p$, $0 < p < 1$, in terms of the modulus of smoothness of arbitrary order $\omega_r(f, n^{-1})_p$, was found by Storozhenko and Oswald in [1] (see also [2]). In Sec. 1 of the present paper, we formulate a theorem (see Theorem 1) that gives a condition under which $E_n(f)_p$ can easily be estimated from below. This theorem is a generalization of the corresponding Rathore’s result [3] to the case of the spaces $L_p(\mathbb{T})$, $0 < p < 1$.

We also consider the problem of extension of functions with preservation of the modulus of smoothness. In the spaces $L_p$, $p \geq 1$, theorems on extension of functions were proved by Dzyadyk [4] and Besov [5] (see also Theorems 4.1 in [6] and 4.6.12 in [7]). In Sec. 2, we prove a theorem on extension of functions from $L_p$, $0 < p < 1$, from a segment to a wider segment and to the entire numerical line with preservation of properties of the modulus of smoothness (see Theorems 2 and 3).

In Sec. 3, we consider Fourier multipliers. The properties of Fourier multipliers in the spaces $L_p(T)$, $p \geq 1$, are well studied (see, e.g., [8], Chap. 4, Sec. II; [9], Chap. 16; and [7], Chap. 7). We consider multipliers in $L_p(T)$ for $0 < p < 1$ and give their complete description (see Theorem 4).

1. In the present paper, the letter $C$ denotes positive constants that depend on the indicated parameters. The constants $C$ may be different even in the same line.

We now give several auxiliary statements. The two properties of the modulus of smoothness presented below hold in both periodic and nonperiodic cases (see, e.g., [6], Chap. 12, Sec. 5 or [7], Chap. 4). Let $0 < p < 1$, $k, r \in \mathbb{N}$ ($k > r$), $h > 0$, and $\lambda > 0$. Then

$$\omega_k(f, h)_p \leq 2^{k-r} \omega_r(f, h)_p \leq 2^r \|f\|_p,$$

(1)

$$\omega_r(f, \lambda h)_p \leq \lambda^{\frac{1}{p}-1}(1 + \lambda)^{\frac{1}{p}+r-1} \omega_r(f, h)_p.$$

(2)

**Theorem A** [1, 2]. Suppose that $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $n, r \in \mathbb{N}$. Then

$$E_{n-1}(f)_p \leq C \omega_r \left( f, \frac{1}{n} \right)_p,$$

where the constant $C$ depends only on $r$ and $p$.

A converse theorem is also known.

**Theorem B** [10]. Suppose that $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $n, r \in \mathbb{N}$. Then

$$\omega_r \left( f, \frac{1}{n} \right)_p \leq C \frac{1}{n^r} \left\{ \sum_{k=0}^{n} (k+1)^{rp-1} E_k(f)_p^p \right\}^{\frac{1}{p}},$$

where the constant $C$ depends only on $r$ and $p$.

Our aim is to prove the following statement:

**Theorem 1.** Suppose that $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $r \in \mathbb{N}$. For the existence of a constant $L > 0$ such that

$$\omega_r \left( f, \frac{1}{n} \right)_p \leq L E_{n-1}(f)_p \quad \text{for all } n \in \mathbb{N},$$

(3)