SEMIPERFECT IPRI-RINGS AND RIGHT BÉZOUT RINGS

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We present a survey of some results on ipri-rings and right Bézout rings. All these rings are generalizations of principal ideal rings. From the general point of view, decomposition theorems are proved for semiperfect ipri-rings and right Bézout rings.

1. Introduction

Recall that a principal right ideal ring is a ring with identity 1 ≠ 0 in which every right ideal is principal (A. W. Goldie [8] called this ideal a pri-ring). A principal left ideal ring (a pli-ring) can be defined similarly. The properties of pli-rings were considered in [15].

A principal ideal ring is a ring which is both a principal right ideal ring and a principal left ideal ring. A ring A with the Jacobson radical R is a primary ring if A/R is a simple Artinian ring.

As one of the main examples of principal ideal rings, we can mention the ring Z of all integers. It is well-known that every commutative principal ideal ring is a finite direct sum of rings which either are integral domains or are completely primary (see [33], Chap. 4). A similar theorem was proved by K. Asano for the case of noncommutative Artinian rings. In [1], he proved that each ring of this sort is a finite direct sum of primary rings.

Goldie considered the structure of pri-rings. In [8], he proved the following main theorems.

Theorem A. A pri-ring without nilpotent ideals is a finite direct sum of prime pri-rings.

Theorem B. A prime pri-ring is a complete matrix ring Kn, where K is a right Noetherian integral domain.

Theorem C. A left Noetherian pri-ring is a finite direct sum of pri-rings each of which is either a prime ring or a primary ring.

J. C. Robson [25] considered wider classes of pri-rings and called them ipri-rings and ipli-rings. An ipri-ring (ipli-ring) is a ring in which every two-sided ideal is a principal right (left) ideal. A ring A with nilpotent radical W is called W-simple if A/W is a simple ring. Robson extended several results concerning pri-rings proved by Goldie. In particular, he proved the following theorems:

Theorem 1 [25]. A Noetherian ipri-ring is a finite direct sum of ideals each of which is a Noetherian ipri-ring and is either prime or W-simple. A Noetherian ipri-ring has a (right and left) quotient ring which is an Artinian pri-ring.

Theorem 2 [25]. If A is a Noetherian ipri- and ipli-ring, then the multiplication of ideals in A is commutative, A is the direct sum of rings each of which is prime or W-simple, and every proper ideal in each of these rings is a unique product of maximal ideals.
Note that earlier the one-sided Artinian ipri- and ipli-rings were considered by N. Jacobson [14].

**Theorem I** ([14], Theorem 37). If \( A \) is a ring with identity satisfying the descending chain condition on one-sided ideals and every two-sided ideal of \( A \) is a principal right ideal and a principal left ideal, then \( A \) is the direct sum of two-sided ideals which are primary rings with the indicated properties.

In [9] (Sec. 12.2), a similar theorem was proved for semiperfect rings.

**Definition 1.1.** A ring \( \mathcal{O} \) (not necessarily commutative) is called a principal ideal domain if it has no zero divisors and all its right and left ideals are principal.

**Theorem 1.1** ([9], Sec. 12.2). Let \( A \) be a semiperfect ring such that every two-sided ideal in \( A \) is both a right principal ideal and a left principal ideal. Then \( A \) is a principal ideal ring isomorphic to the direct product of finitely many full matrix rings over Artinian uniserial rings and local principal ideal domains. Conversely, all rings of this sort are semiperfect principal ideal rings.

Another generalization of pri-rings are right Bézout rings.

**Definition 1.2.** A ring is called right (resp., left) Bézout ring if every finitely generated right (left) ideal of this ring is principal. A ring which is both a right Bézout ring and a left Bézout ring is called a Bézout ring. A Bézout domain is an integral domain in which every finitely generated ideal is principal.

A pri-ring is obviously a right Bézout ring. In a certain sense, a right Bézout ring is a non-Noetherian analog of a pri-ring. On the other hand, the fact that any right ideal in a right Noetherian ring is finitely generated immediately yields the following assertion:

**Proposition 1.1.** A right Noetherian ring is a right Bézout ring if and only if it is a pri-ring.

The main examples of commutative Bézout domains that are not principal ideal domains (PID) and are not Noetherian can be described as follows:

1. The ring \( \mathcal{O}(D) \) of all functions in a single complex variable holomorphic in a domain \( D \) of the complex plane \( \mathbb{C} \).

2. The ring of holomorphic functions given in the entire complex plane \( \mathbb{C} \).

3. The ring of all algebraic integers.

First, the properties of the ring \( \mathcal{O}(D) \) of all functions of a single complex variable holomorphic in a domain \( D \) of the complex plane \( \mathbb{C} \) were studied by J. H. M. Wedderburn in 1915 [29]. He considered the problem of reducing the matrix whose coefficients are functions from the ring \( \mathcal{O}(D) \) to an equivalent diagonal matrix. In particular, he proved the following main lemma:

**Lemma 1.1** (Wedderburn [29]). Let \( f, g \in \mathcal{O}(D) \) be two functions holomorphic in a domain \( D \subset \mathbb{C} \) and relatively prime (i.e., these functions have no common zeros in \( D \)). Then there exist two functions \( p, q \in \mathcal{O}(D) \) holomorphic in \( D \) and such that

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pf + qg = 1.
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