ON THE RESTRICTED PROJECTIVE DIMENSION OF COMPLEXES

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We study the restricted projective dimension of complexes and give some new characterizations of the restricted projective dimension. In particular, it is shown that the restricted projective dimension can be computed in terms of the so-called restricted projective resolutions. As applications, we get some results on the behavior of the restricted projective dimension under the change of rings.

Introduction

It is well known that the classical homological dimensions — projective, flat, and injective — are defined in terms of resolutions. However, they can also be computed in terms of vanishing of appropriate derived functors. Thus, the flat dimension of an \(R\)-module \(M\) can be computed as follows:

\[
\text{fd}_R(M) = \sup \{ i \in \mathbb{N}_0 | \text{Tor}^R_i(T, M) \neq 0 \text{ for some module } T \}.
\]

The restricted flat dimension was defined solely in terms of vanishing of the derived functor \(\text{Tor}\) over some classes of test modules restricted to assure their automatic finiteness over commutative Noetherian rings of finite Krull dimension (see [3]). More precisely, the restricted flat dimension of an \(R\)-module \(M\) is denoted by \(\text{Rfd}_R M\) and defined as follows:

\[
\text{Rfd}_R(M) = \sup \{ i \in \mathbb{N}_0 | \text{Tor}^R_i(T, M) \neq 0 \text{ for some module } T \text{ with } \text{fd}_R(T) < \infty \}.
\]

Christensen, Foxby, and Frankild [3] further studied the restricted flat dimension of complexes and gave a number of interesting properties. Thus, they showed that the restricted flat dimension is finite for any homologically bounded complex over commutative Noetherian rings of finite Krull dimension and can be regarded as a refinement of both flat and Gorenstein flat dimensions. Sharif and Yassemi [4] studied the behavior of the restricted flat dimension under the change of rings and generalized some classical results.

Let \(X\) be a complex of \(R\)-modules homologically bounded below. The restricted projective dimension of \(X\), denoted by \(\text{Rpd}_R X\), was defined by Christensen, Foxby, and Frankild in [3]. They showed that this dimension is also finite for any homologically bounded complex over commutative Noetherian rings of finite Krull dimension. In the present paper, we give some new characterizations of the restricted projective dimension of complexes, which show that the restricted projective dimension can be computed in terms of the so-called restricted projective resolutions (see Theorem 2.1).

**Theorem A.** Let \(X\) be a complex homologically bounded below and let \(n \in \mathbb{Z}\). Consider the following conditions:

1. \(\text{Rpd}_R X \leq n\).

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(2) $X$ is equivalent to a bounded complex $P$ of restricted projective $R$-modules with
\[ \sup \{ i \in \mathbb{Z} \mid P_i \neq 0 \} \leq n \]
and $P$ can be chosen such that $P_l = 0$ for $l < \inf X$.

(3) $H_i(R \text{Hom}_R(X, T)) = 0$ for any $i < -n$ and any $R$-module $T$ with $\text{id}_R(T) < \infty$.

(4) $\sup X \leq n$ and $C_n(P)$ is a restricted projective $R$-module, whenever $P$ is a complex of restricted projective $R$-modules bounded below, which is equivalent to $X$.

(5) $-\inf(R \text{Hom}(X, U)) + \inf U \leq n$ for any nonexact complex $U$ with $\text{id}_R U < \infty$.

Then (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\iff$ (5). If, in addition, $X$ is homologically degree-wise finite, then all statements presented above are equivalent.

As an application of this theorem, we get the following result on the behavior of the restricted projective dimension under the change of rings (see Propositions 2.4 and 2.5).

**Theorem B.** Let $\varphi: R \to S$ be a homomorphism of rings and let $X$ be a degree-wise finite complex of $S$-modules homologically bounded below. Then the following statements hold:

1. If $Y$ is a complex of $R$-modules with $\text{fd}_R Y < \infty$ homologically bounded below, then
\[
\text{Rpd}_R(X \otimes_R^L Y) \leq \text{Rpd}_S X + \text{Rpd}_R Y + \text{Rpd}_R S
\]
and
\[
\text{Rpd}_S(X \otimes_S^L Y) \leq \text{Rpd}_S X + \text{Rfd}_R S + \text{sup} Y + \text{dim} S.
\]

2. If $Y$ is a complex of $S$-modules with $\text{fd}_S Y < \infty$ homologically bounded below, then
\[
\text{Rpd}_R(X \otimes_S^L Y) \leq \text{Rpd}_S X + \text{Rpd}_R Y.
\]

1. Preliminaries

We begin with some notation and terminology taken from [2] and used throughout the paper.

1.1. A complex $\ldots \to X_1 \xrightarrow{\delta^X_1} X_0 \xrightarrow{\delta^X_0} X_{-1} \to \ldots$ of $R$-modules is denoted by $(X, \delta^X)$ or simply by $X$. We frequently (and without special warnings) identify $R$-modules with complexes concentrated in degree 0. A complex $X$ is bounded above (resp., bounded below, or bounded) if $X_n = 0$ for $n \gg 0$ (resp., $n \ll 0$, or $|n| \gg 0$). The $n$th boundary (resp., cycle, homology) of $X$ is defined as $\text{Im} \delta^X_n$ (resp., $\text{Ker} \delta^X_n$, $\text{Ker} \delta^X_n/\text{Im} \delta^X_{n+1}$) and denoted by $B_n(X)$ (resp., $Z_n(X)$, $H_n(X)$). A complex $X$ is homologically bounded above (resp., homologically bounded below, or homologically bounded) if the homology complex $H(X)$ is bounded above (resp., bounded below, or bounded). We use the notation $C_n(X)$ for the cokernel of the differential $\delta^X_{n+1}$. The soft truncations of $X$ at $n$ are the complexes
\[
X_{\leq n} \equiv 0 \to C_n(X) \xrightarrow{\delta^X_n} X_{n-1} \xrightarrow{\delta^X_{n-1}} X_{n-2} \to \ldots
\]