GLOBAL WEAK SOLUTIONS FOR THE WEAKLY DISSIPATIVE
$\mu$-HUNTER–SAXTON EQUATION

J. Liu

UDC 517.9

The paper deals with the global existence of weak solutions for a weakly dissipative $\mu$-Hunter–Saxton equation. The problem is analyzed by using smooth data approximating the initial data and Helly’s theorem.

1. Introduction

Recently, Khesin, et al. [7] deduced and studied the following the $\mu$-Hunter–Saxton equation (also called the $\mu$-Camassa–Holm equation):

$$
\mu(u)_t - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx},
$$

which describes the evolution of rotators in liquid crystals with external magnetic and self-interaction. Here, $u(t, x)$ is a time-dependent function on the unit circle $S = \mathbb{R}/\mathbb{Z}$ and

$$
\mu(u) = \int_{S} u dx
$$

denotes its mean. The $\mu$-Hunter–Saxton equation lies in the midway between the periodic Hunter–Saxton and Camassa–Holm equations. Moreover, this equation describes the geodesic flow on $D^s(S)$ with the right-invariant metric given at the identity by the inner product [7]

$$
(u, v) = \mu(u)\mu(v) + \int_{S} u_x v_x dx.
$$

The Cauchy problem of the $\mu$-Hunter–Saxton equation is extensively studied. It has been shown that the $\mu$-Hunter–Saxton equation is locally well posed [7] with the initial data $u_0 \in H^s(S)$, $s > \frac{3}{2}$. It is of interest that it has global strong solutions [7] and also blow-up solutions in finite time [3, 5, 7] with a different class of initial profiles in the Sobolev spaces $H^s(S)$, $s > \frac{3}{2}$. On the other hand, it has global dissipative weak solutions in $H^1(S)$ [15]. Moreover, the $\mu$-Hunter–Saxton equation admits both the periodic one-peakon solution and the multi-peakons [7, 9].

In general, it is difficult to avoid energy dissipation mechanisms in the real world. Hence, it is reasonable to study the model with energy dissipation. In [4] and [13], the authors discussed different aspects of the energy...
dissipative KdV equation. The weakly dissipative Camassa–Holm equation and weakly dissipative Degasperis–Procesi equation were studied in [17, 19] and [2, 6, 18, 20] respectively. Recently, Wei and Yin [16] discussed the global existence and blow-up phenomena for the weakly dissipative periodic Hunter–Saxton equation.

In the present paper, we discuss the global existence of weak solutions of the following weakly dissipative $\mu$-Hunter–Saxton equation:

\[ y_t + uy_x + 2u_x y + \lambda y = 0, \quad t > 0, \quad x \in \mathbb{R}, \]

\[ y = \mu(u) - u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \]

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \]

\[ u(t, x + 1) = u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}, \]  \hspace{1cm} (1.1)

or, in the equivalent form,

\[ \mu(u)_t - u_{txx} + 2\mu(u)u_x - 2u_x u_{xx} - uu_{xxx} + \lambda(\mu(u) - u_{xx}) = 0, \quad t > 0, \quad x \in \mathbb{R}, \]

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \]

\[ u(t, x + 1) = u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}. \]  \hspace{1cm} (1.2)

Here, the constant $\lambda$ is assumed to be positive and $\lambda y = \lambda(\mu(u) - u_{xx})$ is a weakly dissipative term. The Cauchy problem (1.1) has been recently discussed in [10]. The author established the local well-posedness, derived the precise blow-up scenario for Eq. (1.1), and proved that Eq. (1.1) has global strong solutions and also finite-time blow-up solutions. However, the existence of global weak solutions to Eq. (1.1) has not been studied yet. The aim of the present paper is to establish a global existence result for weak solutions of Eq. (1.1).

Throughout the paper, by $*$ we denote the operation of convolution. Let $\| \cdot \|_Z$ denote the norm in the Banach space $Z$ and let $\langle \cdot, \cdot \rangle$ denote the $H^1(S)$, $H^{-1}(S)$ duality bracket. Let $M(S)$ be the space of Radon measures on $S$ with bounded total variation and let $M^+(S)$ ($M^-(S)$) be the subset of $M(S)$ with positive (negative) measures. Finally, we write $BV(S)$ for the space of functions with bounded variation, where $\nabla(f)$ is the total variation of $f \in BV(S)$.

Prior to giving the precise statement of our main result, we introduce the definition of a weak solution of problem (1.2).

**Definition 1.1.** A function

\[ u(t, x) \in C(\mathbb{R}^+ \times S) \cap L^\infty(\mathbb{R}^+; H^1(S)) \]

is said to be an admissible global weak solution of (1.2) if $u$ satisfies the equations in (1.2) and $z(t, \cdot) \to z_0$ as $t \to 0^+$ in a sense of distributions on $\mathbb{R}^+ \times \mathbb{R}$. Moreover,

\[ \mu(u(t)) = \mu(u_0)e^{-\lambda t} \quad \text{and} \quad \| u_x(t, \cdot) \|_{L^2(S)} = e^{-\lambda t}\| u_0, x \|_{L^2(S)}. \]

The main result of this paper can be formulated as follows: