WE PROPOSE A NEW APPROACH TO THE CLASSICAL MEAN-VALUE THEOREM IN WHICH TWO MEAN VALUES ARE USED INSTEAD OF ONE. THIS APPROACH IS OF ESPECIAL IMPORTANCE FOR COMPLEX FUNCTIONS BECAUSE THERE ARE NO AVAILABLE THEOREMS OF THIS KIND FOR THESE FUNCTIONS.

1. REAL FUNCTIONS

The classical mean-value theorem for a real function \( f(x), \ x \in [a, b] \), states that if the function \( f \) is differentiable on the interval \((a, b)\), then there exists a point \( c \) of this interval, \( a < c < b \), at which the following equality is true:

\[
\frac{f(b) - f(a)}{b-a} = f'(c).
\]

The following statement is proved in [1] for an arbitrary continuous function \( f \):

**Mean-Value Theorem.** Let \( f(x), \ x \in [a, b], \) be a continuous function. Then the following three (generally speaking, consistent) versions are true:

(i) there exists a point \( c, \ a < c < b, \) for which the following equality is true:

\[
\frac{f(b) - f(a)}{b-a} = \delta(c),
\]

where \( \delta(c) \) is a certain right derived number of the function \( f \);

(ii) the set \( E(x: f'(x) = \infty) \) is nonempty and its image \( f(E) \) has a positive measure: \( \text{mes} \ f(E) > 0; \)

(iii) \( f \) is an \( ACG \)-function on \([a, b]\) and

\[
\frac{f(b) - f(a)}{b-a} = \frac{1}{b-a} \int_{a}^{b} f'_{as}(t)dt,
\]

where, in the general case, the integral on the right-hand side is the Denjoy integral in a wide sense.
Recall that a number $A$ is called a derived number of the function $f$ at the point $x_0 \in [a, b]$ if there exists a sequence $h_n \to 0$ such that
\[
\lim_{n \to \infty} \frac{f(x_0 + h_n) - f(x_0)}{h_n} = A;
\]
moredver, $A$ is called a right derived number if all $h_n > 0$ and a left derived number if $h_n < 0$.

We now present another formulation of the mean-value theorem that does not contain infinite derivatives but uses a couple of mean values.

We prove the following theorem:

**Theorem 1.** Let $f(x), x \in [a, b]$, be an arbitrary finite function. If, at every point $x$, except at most a countable set of points, the right derived numbers can be equal only to infinity or to zero, then, in $[a, b]$, one can find a dense set of the intervals of constancy of the function $f$.

**Proof.** We introduce the following sets:

\[
A^+_n = \left\{ x : \frac{f(x + h) - f(x)}{h} \geq 1 \quad \text{for} \quad 0 < h \leq \frac{1}{n} \right\},
\]

\[
A^-_n = \left\{ x : \frac{f(x + h) - f(x)}{h} \leq -1 \quad \text{for} \quad 0 < h \leq \frac{1}{n} \right\},
\]

\[
B_n = \left\{ x : \frac{f(x + h) - f(x)}{h} \leq 1 \quad \text{for} \quad 0 < h \leq \frac{1}{n} \right\}
\]

and take an arbitrary segment $[\alpha, \beta] \subset [a, b]$. By the condition of the theorem, we have

\[
[\alpha, \beta] = A \cup B,
\]

where

\[
A = E\{x : f'(x) = \infty\}, \quad B = \{x : f'(x) = 0\} \quad \text{and} \quad A = \bigcup_n A^+_n \cup \left( \bigcup_n A^-_n \right), \quad B = \bigcup_n B_n.
\]

If the set $A$ is not of the first category, then one of the sets $A^+_n$ is dense on the segment $[\alpha', \beta'] \subset [\alpha, \beta]$ whose length can be smaller than $1/h$. The definition of $A^+_n$ immediately implies that $f$ is also strictly monotone everywhere in this segment with derived numbers whose absolute values are greater than $+1$. However, this function is differentiable almost everywhere on $[\alpha', \beta']$ with derivatives different from $\infty$ and $0$.

Hence, everywhere on $[\alpha, \beta]$, the set $B$ must be of the second category and, on the partial segment $[\alpha', \beta']$, $f$ is a Lipschitz function without infinite derivatives. Thus, by the condition, there exist only zero derived numbers, which means that the function $f$ is constant on $[\alpha', \beta']$.

Since the segment $[\alpha, \beta] \subset [a, b]$ is arbitrary, we arrive at the statement of the theorem.

It can be proved that, under the conditions of Theorem 1, the complement to the obtained system of intervals of constancy is a countable nowhere dense set on $[a, b]$. However, this result is not used in what follows.

For a function $f$ strictly monotone on $[a, b]$, this statement means that there exists a set of points dense on $[a, b]$ at which finite right derived numbers exist and are different from zero.\(^1\)

\(^1\) This statement is nontrivial because a strictly monotone function may have zero derived numbers both almost everywhere and on a set of the second category, infinite derivatives also on a set of the second category, etc.