BRIEF COMMUNICATIONS

ON THE SOLVABILITY OF A FOURTH-ORDER OPERATOR-DIFFERENTIAL EQUATION WITH MULTIPLE CHARACTERISTIC

A. R. Aliev

In the Sobolev-type space with exponential weight, we obtain sufficient conditions for the well-posed and unique solvability on the entire axis of a fourth-order operator-differential equation whose main part has a multiple characteristic. We establish estimates for the norms of the operators of intermediate derivatives related to the conditions of solvability. In addition, we deduce the relationship between the exponent of the weight and the lower bound of the spectrum of the main operator appearing in the principal part of the equation. The obtained results are illustrated by an example of a problem for partial differential equations.

It is known that theories are developed in analyzing specific model problems. In this sense, the theory of operator-differential equations is not an exception. In [1–3], one can find a comprehensive presentation of the results obtained in the field of operator-differential equations (mainly of the first and second orders). From the viewpoint of applications to various problems of mechanics, it is also necessary to mention the works devoted to the fourth-order operator-differential equations (see [4] and the references therein).

In a separable Hilbert space \( H \), we consider a fourth-order operator-differential equation

\[
\left( -\frac{d}{dx} + A \right) \left( \frac{d}{dx} + A \right)^3 u(x) + \sum_{j=1}^{4} A_j \frac{d^{4-j} u(x)}{dx^{4-j}} = f(x), \quad x \in \mathbb{R} = (-\infty, +\infty),
\]

where \( f(x) \) and \( u(x) \) are \( H \)-valued functions, \( A \) is a self-adjoint positive-definite operator with lower boundary of the spectrum \( \lambda_0 \) (\( A = A^* \geq \lambda_0 E \) (\( \lambda_0 > 0 \)), where \( E \) is the identity operator), and \( A_j, j = 1, 2, 3, 4 \), are linear, generally speaking, unbounded operators. It is clear that the principal part of Eq. (1) has a multiple characteristic.

Note that equations of the form (1) are encountered in applications, in particular, in the problems of stability of plates made of plastic materials.

We introduce the following Hilbert spaces with weight \( e^{-\kappa x/2}, \kappa \in \mathbb{R} \):

\[
L_{2,\kappa}(\mathbb{R}; H) = \left\{ u(x) : \|u\|_{L_{2,\kappa}(\mathbb{R}; H)} = \left( \int_{-\infty}^{+\infty} \|u(x)\|^2_H e^{-\kappa x} dx \right)^{1/2} < +\infty \right\},
\]

\[
W_{2,\kappa}^n(\mathbb{R}; H) = \left\{ u(x) : \|u\|_{W_{2,\kappa}^n(\mathbb{R}; H)} = \left( \int_{-\infty}^{+\infty} \left( \left| \frac{d^n u(x)}{dx^n} \right|^2_H + \|A^n u(x)\|^2_H \right) e^{-\kappa x} dx \right)^{1/2} < +\infty \right\}, \quad n \geq 1.
\]
In the present paper, under certain algebraic conditions imposed on the operator coefficients, we prove that, for any \( f(x) \in L_{2, \kappa}(R; H) \), Eq. (1) has a unique solution \( u(x) \in W^4_{2, \kappa}(R; H) \).

Note that the well-posed and unique solvability of the boundary-value problem on the semiaxis for Eq. (1) is studied in [4]. The Fredholm property of the boundary-value problems on the semiaxis and on a finite interval is considered for equations of this type in [5].

**Definition.** If, for any \( f(x) \in L_{2, \kappa}(R; H) \), there exists a vector function

\[ u(x) \in W^4_{2, \kappa}(R; H) \]

satisfying Eq. (1) almost everywhere and, in addition, the inequality

\[ \|u\|_{W^4_{2, \kappa}(R; H)} \leq \text{const} \|f\|_{L_{2, \kappa}(R; H)} \]

is true, then this function is called a regular solution of Eq. (1) and Eq. (1) is called regularly solvable.

First, we study Eq. (1) in the case where \( A_j = 0, \ j = 1, 2, 3, 4 \).

Consider a polynomial operator bundle

\[ P_0(\mu; A) = (-\mu E + A)(\mu E + A)^3. \]

By \( P_0 \), we denote an operator acting from the space \( W^4_{2, \kappa}(R; H) \) into the space \( L_{2, \kappa}(R; H) \) as follows:

\[ P_0 u(x) \equiv P_0 \left( \frac{d}{dx}; A \right) u(x), \quad u(x) \in W^4_{2, \kappa}(R; H). \]

The following theorem is true:

**Theorem 1.** Let \( |\kappa| < 2\lambda_0 \). Then the operator \( P_0 \) realizes an isomorphism from the space \( W^4_{2, \kappa}(R; H) \) into the space \( L_{2, \kappa}(R; H) \).

**Proof.** By using the notation introduced above, we rewrite Eq. (1) for \( A_j = 0, \ j = 1, 2, 3, 4 \), in the form

\[ P_0 \left( \frac{d}{dx}; A \right) u(x) = f(x), \quad (2) \]

where \( f(x) \in L_{2, \kappa}(R; H) \) and \( u(x) \in W^4_{2, \kappa}(R; H) \).

We now set \( u(x) = v(x)e^{\kappa x/2} \). Thus, Eq. (2) takes the form

\[ P_0 \left( \frac{d}{dx} + \frac{\kappa}{2}; A \right) v(x) = g(x), \quad (3) \]

where \( v(x) \in W^2_{2}(R; H) \), \( g(x) = f(x)e^{-\kappa x/2} \in L_{2}(R; H) \) and \( W^4_{2}(R; H) = W^4_{2,0}(R; H) \), \( L_{2}(R; H) = L_{2,0}(R; H) \) (for details concerning the spaces \( L_{2}(R; H) \) and \( W^4_{2}(R; H) \), see [6]).

Since the mapping

\[ v(x) \rightarrow u(x)e^{-\kappa x/2} \]

is an isomorphism between the spaces \( W^4_{2}(R; H) \) and \( W^4_{2, \kappa}(R; H) \), to complete the proof of the theorem, it suf-