HISTORY OF THE APPEARANCE OF INFINITE-DIMENSIONAL ANALYSIS AND ITS DEVELOPMENT IN UKRAINE

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We present a brief survey of the development of functional analysis in Ukraine and the problems of infinite-dimensional analysis posed and solved for thousands of years, which laid the foundations of this branch of mathematics.

The monographs “Théorie des Operations Linéaires” by S. Banach and “Linear Transformations in Hilbert Space and Their Applications to Analysis” by M. Stone published in 1932 marked the appearance of one of the main branches of contemporary mathematics, namely, of the infinite-dimensional (functional) analysis aimed at the investigation of functions $y = f(x)$ in the case where, unlike the classical analysis, at least one of the quantities $x$ or $y$ varies in an infinite-dimensional space. Nevertheless, it is worth noting that problems of infinite-dimensional analysis were posed much earlier and solved for thousands of years by numerous mathematicians. Literally, one can say that most famous of these problems laid the foundations for the development of the entire mathematics. Quite often, the solution of some of these problems requires approaches lying beyond the framework of the existing mathematics and, as a final result, leads to the formation of new branches of mathematics. We dwell solely upon the problems dealing with the following two directions: (i) determination of extreme values characterizing the actual regularities of our life; (ii) establishment of harmony between continuous and discrete in the process of development of science. In addition, we present a brief survey of the development of functional analysis in Ukraine.

1. Extremal Problems

Problems of determination of the maximum and minimum values of variables were studied even by ancient scientists. The classic isoperimetric problem is one of the most known problems of this kind. In this problem, it is necessary to determine a curve enclosing the maximum possible area among the closed curves of the same length in a plane. A similar problem was also posed in the three-dimensional space. Simplicios, one of the last representatives of the Plato School of Athens (6th century AD), writes in his comments to Aristotle works (4th century AD): “It has been proved before Aristotle (because he used this as a known fact) and later more completely by Archimedes and Zenodorus that, among the isoperimetric figures, the maximum area is enclosed by a disk and, among the surfaces of the same area figures, the maximum volume is enclosed by a ball.” However, the proof of this fact was not found in the ancient literature. It seems likely that, in this statement, the proof was, in fact, absent. Actually, this proof was obtained only in the 19th century both by analytic and geometric methods. Clearly, this problem belongs to the infinite-dimensional analysis.

From the “The Aeneid” poem by Virgil in which a history attributed to the 7th century BC is presented, we learn about the so-called Dido’s problem (an extreme problem of the infinite-dimensional analysis). Recall that ancient mathematicians also solved numerous other extreme problems encountered both directly in mathematics itself and in the applied problems. Moreover, each of these problems we solved by its own original method. A general method was absent up to the 17th century. Most likely, J. Kepler laid the first foundations of the integral...
and differential calculus in his monograph “Nova Stereometria Doliorum Vinariorum,” Linz (1615) and gave the first general rules of finding the extrema formulated, in 1629, by P. Fermat in the form of the exact theorem for the case where $f(x)$ is a polynomial. Later, I. Newton and G. Leibniz studied the general case, where $f(x)$ is a function of one variable.

The problem of finding the curve of quickest descent along which a body moves under the action of gravity from one point to another for minimum time (the brachistochrone problem) played a very important role in the history of extreme problems. This problem was formulated by Johann Bernoulli in 1696 as a challenge to mathematicians of that time and, in particular, to his older brother Jacob Bernoulli openly making fun of him for the incompetence in mathematics. This problem was solved by Johann Bernoulli and also by Leibniz, Jacob Bernoulli, and Newton. It is worth noting that older Bernoulli offered more original solution than his younger brother. Moreover, he drew attention to the fact (unnoticed by the other researchers) that the problem of finding a curve with the property of minimality or maximality among all curves passing through two given points is a problem of new type, which requires new methods for its solution. Indeed, the set of all parallelograms inscribed in a triangle (or the set of all cylinders inscribed in a ball) depends on a single parameter. Hence, the problem of finding a parallelogram with the maximum area in the first case (Euclidean problem) or a cylinder with maximum volume in the second case (Kepler problem) is reduced to the problem of extremum of a function of one variable. At the same time, for the brachistochrone problem, the set of curves including the extremal curve is infinite-dimensional and, hence, this problem is reduced to the problem of extremum of a function of infinitely many variables. Thus, the solution of this problem led to an unexpected jump in the theory of extreme problems from functions of a single variable to functions of infinitely many variables. The solution of the brachistochrone problem by Johann Bernoulli stimulated the appearance of numerous other similar problems. However, each of these problems was solved by using its own “secret” procedure. For this reason, Johann Bernoulli proposed his disciple L. Euler to find a general approach to the solution of these problems.

In his work “Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, Sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti” (1744), Euler laid the theoretical foundations of a new branch of mathematics called variational calculus. In the cited work, a general approach to the solution of a series of extreme problems from various fields of natural sciences and mathematics was developed for the first time. According to this method, the problems are reduced to finding the maximum (or minimum) of a certain functional of the form

$$F(y) = \int_a^b f(x, y(x), y'(x)) \, dx,$$

where $f$ is a fixed function of three variables, defined on a certain (admissible) class of functions $y(x)$. It was shown that the functional attains its extremal for a function $y(x)$ satisfying the differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0.$$

The solution of the problem is based on the idea of approximation of the extremal by broken lines used, for the first time, by Leibniz for the solution of the brachistochrone problem. This idea was also used by ancient geometers, e.g., to find the areas of some geometric figures. Despite the geometric clearness of this method, it is quite cumbersome and incompletely substantiated up to now.

At that time, Eq. (2) was regarded as convenient for everybody, which seems to be quite clear under the conditions of absence of clear definitions of the main concepts of mathematical analysis, including convergence. The solutions of some extremal problems obtained earlier by using their own specific methods confirmed that Eq. (2) is correct because the functions at which the extrema were attained satisfied this equation. Note that, within