ON \(q\)-CONGRUENCES INVOLVING HARMONIC NUMBERS

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We give some congruences involving \(q\)-harmonic numbers and alternating \(q\)-harmonic numbers of order \(m\). Some of these congruences are \(q\)-analogs of several known congruences.

1. Introduction

For any positive integer \(n\), the \(q\)-integer can be defined as follows:

\[
[n]_q = \frac{1-q^n}{1-q}.
\]

It is easy to see that \(\lim_{q \to 1} [n]_q = n\). Suppose that \(a \equiv b \pmod{p}\). This yields

\[
[a]_q = \frac{1-q^a}{1-q} = \frac{1-q^b + q^b(1-q^{a-b})}{1-q} \equiv \frac{1-q^b}{1-q} = [b]_q \pmod{[p]_q}.
\]

Here and in what follows, every congruence is considered over a polynomial ring \(\mathbb{Z}[q]\) in the variable \(q\) with integral coefficients.

For \(m = 1, 2, 3, \ldots\) and \(n = 0, 1, 2, \ldots\), we define

\[
H_0^{(m)} = 0, \quad H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m} \quad \text{for} \quad n \geq 1
\]

and say that this is a harmonic number of order \(m\). These \(H_n = H_n^{(1)}\) are usually called classical harmonic numbers. Similarly, the alternating harmonic numbers of order \(m\) are given by

\[
I_0^{(m)} = 0, \quad I_n^{(m)} = \sum_{j=1}^{n} \frac{(-1)^j}{j^m} \quad \text{for} \quad n \geq 1.
\]

In the present paper, we define

\[
H_n(q) = \sum_{j=1}^{n} \frac{1}{[j]_q}, \quad \tilde{H}_n(q) = \sum_{j=1}^{n} \frac{q^j}{[j]_q}.
\]
\[
H_n^{(2)}(q) = \sum_{j=1}^{n} \frac{1}{[j]_q^2}, \quad \tilde{H}_n^{(2)}(q) = \sum_{j=1}^{n} \frac{q^j}{[j]_q^2},
\]
\[
H_n^{(3)}(q) = \sum_{j=1}^{n} \frac{1}{[j]_q^3}, \quad \tilde{H}_n^{(3)}(q) = \sum_{j=1}^{n} \frac{q^j}{[j]_q^3},
\]

and
\[
I_n(q) = \sum_{j=1}^{n} \frac{(-1)^j}{[j]_q},
\]
\[
I_n^{(2)}(q) = \sum_{j=1}^{n} \frac{(-1)^j}{[j]_q^2},
\]

where
\[
H_0(q) = \tilde{H}_0(q) = H_0^{(2)}(q) = \tilde{H}_0^{(2)}(q) = H_0^{(3)}(q) = \tilde{H}_0^{(3)}(q) = H_0(q) = I_0(q) = I_0^{(2)}(q) = 0.
\]

They are \(q\)-analogs of harmonic numbers of order \(m\). Hence, we call them \(q\)-harmonic numbers and alternating \(q\)-harmonic numbers of order \(m\).

By using the \(q\)-analog of Glaisher’s congruence, Andrews [1] (Theorem 4) showed that
\[
H_{p-1}(q) \equiv \frac{p-1}{2}(1-q) \pmod {[p]_q}
\]
and
\[
\tilde{H}_{p-1}(q) \equiv \frac{p-1}{2}(q-1) \pmod {[p]_q}.
\]

L. L. Shi and H. Pan obtained (see [6], Theorem 1)
\[
H_{p-1}(q) \equiv \frac{p-1}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2[p]_q \pmod {[p]_q^2}
\]
for each \(p \geq 5\), which is equivalent to
\[
\tilde{H}_{p-1}(q) \equiv \frac{1-p}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2[p]_q \pmod {[p]_q^2}.
\]

Pan established (see [5], Theorem 1.1) that, for each odd prime \(p\), we have
\[
2 \sum_{j=1}^{p-1} \frac{1}{[2j]_q} + 2Q_p(2, q) - Q_p(2, q)^2[p]_q
\]
\[
\equiv \left( Q_p(2, q)(1-q) + \frac{p^2-1}{8}(1-q)^2 \right)[p]_q \pmod {[p]_q^2},
\]
(1.2)