Curvature-Driven PDE Methods for Matrix-Valued Images

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Abstract. Matrix-valued data sets arise in a number of applications including diffusion tensor magnetic resonance imaging (DT-MRI) and physical measurements of anisotropic behaviour. Consequently, there arises the need to filter and segment such tensor fields. In order to detect edge-like structures in tensor fields, we first generalise Di Zenzo’s concept of a structure tensor for vector-valued images to tensor-valued data. This structure tensor allows us to extend scalar-valued mean curvature motion and self-snakes to the tensor setting. We present both two-dimensional and three-dimensional formulations, and we prove that these filters maintain positive semidefiniteness if the initial matrix data are positive semidefinite. We give an interpretation of tensorial mean curvature motion as a process for which the corresponding curve evolution of each generalised level line is the gradient descent of its total length. Moreover, we propose a geodesic active contour model for segmenting tensor fields and interpret it as a minimiser of a suitable energy functional with a metric induced by the tensor image. Since tensorial active contours incorporate information from all channels, they give a contour representation that is highly robust under noise. Experiments on three-dimensional DT-MRI data and an indefinite tensor field from fluid dynamics show that the proposed methods inherit the essential properties of their scalar-valued counterparts.

Keywords: DT-MRI, denoising, segmentation, edge detection, structure tensor, mean curvature motion, self-snakes, active contours

1. Introduction

Curvature-based partial differential equations (PDEs) play an important role in image processing and computer vision. They include a number of interesting PDEs such as mean curvature motion (Alvarez et al., 1992), self-snakes (Sapiro, 2001) and geodesic active contours (Caselles et al., 1995; Kichenassamy et al., 1995). Often it is helpful to study their behaviour by investigating the evolutions of the corresponding level lines. On one hand this links these techniques to level set methods (Dervieux and Thomasset, 1979; Osher and Sethian, 1988; Osher and Fedkiw, 2002; Osher and Paragios, 2003; Sethian, 1999), on the other hand one may interpret these evolutions as steepest descent strategies for minimising interesting energy functionals.

While curvature-based PDEs have been extended in various ways to higher dimensions, surfaces and vector-valued data (see e.g. Kimmel, 2003; Osher and Paragios, 2003; Sapiro, 2001), there are hardly any attempts so far to use them for processing tensor-valued data sets. However, such data sets are becoming increasingly important for three reasons:

1. Novel medical imaging techniques such as diffusion tensor magnetic resonance imaging (DT-MRI) have been introduced (Pierpaoli et al., 1996). DT-MRI is a 3-D imaging method that yields a diffusion tensor in each voxel. This diffusion tensor describes the
diffusive behaviour of water molecules in the tissue. It can be represented by a positive semidefinite $3 \times 3$ matrix in each voxel.

2. Tensors have shown their use as a general tool in image analysis, segmentation and grouping (Granlund and Knutsson, 1995; Medioni et al., 2000). This also includes widespread applications of the so-called structure tensor in fields ranging from motion analysis to texture segmentation; see e.g. (Bigün et al., 1991; Rao and Schunck, 1991).

3. A number of scientific applications require the visualisation and processing of tensor fields. The tensor concept is a common physical description of anisotropic behaviour, especially in solid mechanics and civil engineering (e.g. stress-strain relationships, inertia tensors, diffusion tensors, permittivity tensors).

The search for good smoothing techniques for DT-MRI data and related tensor fields is a very recent research area. Several authors have addressed this problem by smoothing derived expressions such as the eigenvalues and eigenvectors of the diffusion tensor (Coulon et al., 2001; Poupon et al., 1998; Tschumperlé and Deriche, 2002) or rotationally invariant scalar-valued expressions (Parker et al., 2000; Zhukov et al., 2003).

Also for fiber tracking applications, most techniques work on scalar- or vector-valued data (Campbell et al., 2002; Vemuri et al., 2001). Some image processing methods that work directly on the tensor components use linear (Westin et al., 1999) or nonlinear (Hahn et al., 2001) techniques that filter all channels independently, thus performing scalar-valued filtering again. Nonlinear variational methods for matrix-valued filtering with channel coupling have been proposed both in the isotropic (Tschumperlé and Deriche, 2002) and in the anisotropic setting (Weickert and Brox, 2002). Related nonlinear diffusion methods for tensor-valued data have led to the notion of a nonlinear structure tensor (Weickert and Brox, 2002) that has been used for optic flow estimation (Brox and Weickert, 2002), texture discrimination and tracking (Brox et al., 2003). Recently also tensorial generalisations of median filtering (Welk et al., 2003), morphological methods (Burgeth et al., 2004), and Mumford–Shah segmentations (Wang and Vemuri, 2004) have been studied. We are, however, not aware of any attempts to generalise curvature-based PDEs to the tensor setting.

The goal of the present paper is to introduce three curvature-based PDEs for analysing and processing two- and three-dimensional tensor fields. They can be regarded as tensor-valued extensions of mean curvature motion, self-snakes and geodesic active contours. The key ingredient for this generalisation is the use of a structure tensor for matrix-valued data.

It should be mentioned that, depending on a certain image processing task at hand, numerous other PDEs have been proposed, including for instance area-preserving flows (Huisken, 1984; Gage, 1986; Sapiro and Tannenbaum, 1995) or methods for processing texture flows (Ben-Shahar and Zucker, 2003; Kimmel and Sochen, 2002; Sapiro, 2001; Weickert, 1998). In our paper we restrict ourselves to three representatives for curvature-based PDEs, since we believe that these models serve as proof-of-concept that motivates to study also other extensions.

Our paper is organised as follows. In Section 2 we introduce the generalised structure tensor for matrix fields. It is then used in Section 3 for designing a 2-D mean curvature type evolution of tensor-valued data. In this section we also derive a variational formulation of tensorial mean curvature motion that is in accordance with its scalar counterpart. Modifying tensor-valued mean curvature motion by a suitable edge stopping function leads us to tensor-valued self-snakes. They are discussed in Section 4. In Section 5, we use the self-snake model in order to derive geodesic active contour models for tensor fields. Three-dimensional extensions of mean curvature motion, self-snakes and geodesic active contours are presented in Section 6, where we also prove that tensorial mean curvature motion preserves positive semidefiniteness of the input data. Algorithmic details are sketched in Section 7, and two- as well as three-dimensional experiments are presented in Section 8. The paper is concluded with a summary in Section 9.

A preliminary, shorter version of our paper has been presented at VLSM 2003 (Federn et al., 2003). The present paper extends this work substantially: It derives three-dimensional results, it presents interpretations in terms of energy functionals that are minimised, and it shows additional experiments with indefinite tensor fields from fluid dynamics.

2. Structure Analysis of Tensor-Valued Data

In this section we generalise the concept of an image gradient to the tensor-valued setting. This may be regarded as a tensor extension of Di Zenzo's and