Smoothing Effects for the Classical Solutions to the Landau-Fermi-Dirac Equation*

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Abstract. The smoothness of the solutions to the full Landau equation for Fermi-Dirac particles is investigated. It is shown that the classical solutions near equilibrium to the Landau-Fermi-Dirac equation have a regularizing effects in all variables (time, space and velocity), that is, they become immediately smooth with respect to all variables.

Keywords. Landau-Fermi-Dirac equation, Classical solutions, Smoothing effect

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1 Introduction and the Statement of Our Main Results

In this paper, we study the regularity of the solutions to the Landau-Fermi-Dirac (LFD) equation for the Pauli exclusion principle which reads

$$\begin{align*}
\partial_t f + v \cdot \nabla_x f &= Q(f, f), \quad t > 0, \\
\quad f(0, x, v) &= f_0(x, v),
\end{align*}$$

(1.1)

where \(f(t, x, v) \geq 0\) is the spatially periodic distribution function for the particles at time \(t \geq 0\), with spatial coordinates \(x = (x_1, x_2, x_3) \in [-\pi, \pi]^3 = T^3\) and velocity \(v = (v_1, v_2, v_3) \in \mathbb{R}^3\). \(Q(f, f)\) is the nonlinear collision operator defined by

$$Q(f, f) = \sum_{i,j=1}^3 \partial_{v_i} \int_{\mathbb{R}^3} \psi^{ij}(v-u)[(1 - f(u))f(u)\partial_{v_j} f(v) - (1 - f(v))f(v)\partial_{v_j} f(u)]du.$$

The non-negative matrix \(\psi\) is

$$\psi^{ij}(v) = \left\{ \delta^{ij} - \frac{v_i v_j}{|v|^2} \right\}|v|^{\gamma+2}.$$

Here, \(\gamma\) is a parameter leading to the standard classification of the hard potential (\(\gamma > 0\)), Maxwellian molecule (\(\gamma = 0\)) or soft potential (\(\gamma < 0\)) (cf. [11]). In particular, \(\gamma = -3\) corresponds to the Coulomb interaction in plasma physics. We recall that the Coulomb potential is, however, the only one to have a physical relevance. In this paper, we restrict our discussion to the case \(-3 \leq \gamma < -2\).

The Landau equation, which was proposed by Landau in 1936, was formally obtained in a singular limit of the Boltzmann equation (cf. [2, 6, 12]). Sometimes, the quantum effects such as...
the Pauli exclusion principle would be taken into account and both the Boltzmann and Landau equations have to be modified (cf. [6, 10, 14, 19]). Among them, the Boltzmann-Fermi-Dirac (BFD) equation and the LFD equation are two typical models. We mention that using a new sequence of cross-sections in the BFD operator and taking a limit (grazing collision limit) lead to the quantum LFD operator (cf. [10, 17]). While the classical Landau equations are the subject of several papers [6, 9, 11, 13, 16, 20] and the references therein, few studies were devoted to the LFD equation. For the LFD equation, a formal derivation from the BFD equation in the grazing collisions limit and a spectral analysis of its linearization near an equilibrium were studied in [10] and [17] respectively. In the spatially homogenous setting, the well-posedness of the Cauchy problem was considered in [3] and the equilibrium states were given in [4]. For the spatially inhomogeneous case, very recently, the global-in-time classical solutions near equilibrium have been constructed in [18].

We are now concerned with the regularity issues. In the spatially homogeneous setting, the regularity of the solutions to the Landau equation was investigated by Arsen’ev-Buryak [2] in the Coulomb case. The instantaneous smoothing effect was shown by Desvillettes-Villani [13], El Safadi [15] and Chen-Li-Xu [7] for not necessarily smooth initial data in the case of hard potentials. In the spatially inhomogeneous setting, recently, Chen-Desvillettes-He [9] and also Alexandre et al. [1] have developed independent machinery to study these general smoothing effects for kinetic equations. We note that the well-posedness results of the above mentioned equations can be found in the references of the corresponding papers.

As far as we know, for the regularity properties of the LFD equation, very few results are available. We would like to mention that Chen [8] got the smoothing effect of Bagland’s weak solutions (cf. [3]) to the spatially homogeneous LFD equation for hard potentials. Our goal in this paper is to obtain the regularity effect of the spatially inhomogeneous LFD equation. We obtain the smoothness in the velocity variable by using energy methods and the dissipative property of the LFD collision operator, where smoothness in the velocity variable was obtained by the elliptic property of the diffusive matrix to the LFD collision operator. Since the LFD nonlinear operator $Q$ can be written as the diffusive operator and some error terms, smoothness in the position variable can be shown by using the classical averaging lemma (cf. [5]). Lastly, we prove the smoothness in the time variable as [9] and deduce the smoothness in all variables by the iterative methods in [1, 8–9]. Although our main results are proved by using the novel idea of [9], there are two main difficulties in this paper. The first new difficulty is due to the complexity of the nonlinear term $f(1 - f)$. The $L^\infty$-norm of $f$ is repeatedly used to overcome this difficulty. The second one is to obtain the lower bound of $\sum_{ij} a_{ij} \xi_i \xi_j$, and unlike the classical Landau equation, the smallness of the solutions in the norm $H^s_{x,v}$ and the Sobolev embedding as well as the velocity splitting method introduced in [13] guarantee the elliptic property of $a_{ij}$. We have to detailedly use the property of the quadratic term $f(1 - f)$ and the Pauli exclusion principle to get the corresponding estimates.

Now we introduce some notations and definitions. For simplicity, we omit the integrating domains $T^3$ and $\mathbb{R}^3$, which correspond to variables $x$ and $v$, respectively. For example, we write $L^2_{x,v}$ instead of $L^2_T(L^2_x(L^2_v(\mathbb{R}^3)))$. For $s \in \mathbb{R}$, we use the standard notation $H^s$ to denote the usual Sobolev space, and use $\dot{H}^s$ to denote the homogeneous Sobolev space. For any integer