AN EXPONENTIAL INEQUALITY FOR AUTOREGRESSIVE PROCESSES IN ADAPTIVE TRACKING*

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Abstract   A wide range of literature concerning classical asymptotic properties for linear models with adaptive control is available, such as strong laws of large numbers or central limit theorems. Unfortunately, in contrast with the situation without control, it appears to be impossible to find sharp asymptotic or nonasymptotic properties such as large deviation principles or exponential inequalities. Our purpose is to provide a first step towards that direction by proving a very simple exponential inequality for the standard least squares estimator of the unknown parameter of Gaussian autoregressive process in adaptive tracking.

Key words  Adaptive tracking, autoregressive process, exponential inequalities, least squares, martingales.

1 Introduction

First of all, we recall the celebrated Hoeffding, Bennett and Bernstein exponential inequalities for sums of bounded independent random variables. We refer the reader to [1], [2] and the excellent survey of McDiarmid[3].

**Theorem 1 (Hoeffding’s inequality)*** Let \((X_n)\) be a sequence of independent random variables such that, for each \(1 \leq k \leq n\), \(a_k \leq X_k \leq b_k\) a.s. for some constants \(a_k < b_k\). If \(S_n = X_1 + X_2 + \cdots + X_n\), then for all \(x \geq 0\),

\[
P(|S_n - E[S_n]| \geq x) \leq 2 \exp\left(-\frac{2x^2}{\sum_{k=1}^{n}(b_k - a_k)^2}\right).
\]

**Theorem 2 (Bennett’s inequality)*** Let \((X_n)\) be a sequence of independent square integrable random variables such that, for each \(1 \leq k \leq n\), \(X_k \leq c\) a.s. for some constant \(c > 0\). If \(S_n = X_1 + X_2 + \cdots + X_n\) and \(V_n\) is the variance of \(S_n\), then for all \(x \geq 0\),

\[
P(S_n - E[S_n] \geq x) \leq \exp\left(-\frac{V_n}{c^2} h\left(\frac{xe}{V_n}\right)\right),
\]

where \(h(x) = (1 + x) \log(1 + x) - x\).

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Theorem 3 (Bernstein’s inequality) Under the assumptions of Theorem 2, we have for all \( x \geq 0 \),
\[
P(S_n - \mathbb{E}[S_n] \geq x) \leq \exp\left(-\frac{x^2}{2(V_n + \frac{3}{2}x)}\right).
\] (3)

Remark 1 One can observe that Bernstein’s inequality immediately follows from (2) as for all \( x \geq 0 \),
\[
h(x) \geq \frac{3x^2}{2(3 + x)}.
\]
Moreover, if \( V_n = \sigma^2 n \) with \( \sigma^2 > 0 \) and \( x = o(n) \), then the upper bound in (3) behaves like \( \exp(-x^2/2\sigma^2 n) \) which coincides with the Gaussian upper bound since by the central limit theorem,
\[
\frac{S_n - \mathbb{E}[S_n]}{\sigma \sqrt{n}} \xrightarrow{L} \mathcal{N}(0, 1).
\]

2 Exponential Inequalities for Martingales

We shall now focus our attention on exponential inequalities for martingales. Let \((M_n)\) be a locally square integrable real martingale adapted to a filtration \( F = (\mathcal{F}_n) \) with initial value \( M_0 = 0 \). The predictable quadratic variation and the total quadratic variation of \((M_n)\) are respectively given by
\[
\langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}[\Delta M_k^2 | \mathcal{F}_{k-1}] \quad \text{and} \quad [M]_n = \sum_{k=1}^{n} \Delta M_k^2,
\]
where \( \Delta M_n = M_n - M_{n-1} \). Hoeffding’s inequality holds for bounded martingale difference sequences by the so-called Azuma-Hoeffding’s inequality\(^4\).

Theorem 4 (Azuma-Hoeffding’s inequality) Let \((M_n)\) be a locally square integrable real martingale such that, for each \( 1 \leq k \leq n \), \( a_k \leq \Delta M_k \leq b_k \) a.s. for some constants \( a_k < b_k \). Then, for all \( x \geq 0 \),
\[
P(|M_n| \geq x) \leq 2 \exp\left(-\frac{2x^2}{\sum_{k=1}^{n} (b_k - a_k)^2}\right). \] (4)

Another result which involves the predictable quadratic variation \( \langle M \rangle_n \) is the famous Freedman’s inequality\(^5\).

Theorem 5 (Freedman’s inequality) Let \((M_n)\) be a locally square integrable real martingale such that, for each \( 1 \leq k \leq n \), \( \Delta M_k \leq c \) a.s. for some constant \( c > 0 \). Then, for all \( x, y > 0 \),
\[
P(M_n \geq x, [M]_n \leq y) \leq \exp\left(-\frac{x^2}{2(y + cx)}\right). \] (5)

Over the last decade, extensive study has been made to establish exponential inequalities for self-normalized martingales relaxing the boundedness assumption on the increments. The most important advance is probably De la Peña’s contribution\(^6\). We shall say that a real martingale \((M_n)\) adapted to \( \mathbb{F} = (\mathcal{F}_n) \) is conditionally symmetric if, for all \( n \geq 1 \), the conditional distribution of \( \Delta M_n \) given \( \mathcal{F}_{n-1} \) is symmetric.

Theorem 6 (De la Peña’s inequality) Let \((M_n)\) be a locally square integrable and conditionally symmetric real martingale. Then, for all \( x, y > 0 \),
\[
P(M_n \geq x, [M]_n \leq y) \leq \exp\left(-\frac{x^2}{2y}\right). \] (6)

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