THE LIMIT THEOREM FOR DEPENDENT RANDOM VARIABLES WITH APPLICATIONS TO AUTOREGRESSION MODELS

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DOI: 10.1007/s11424-011-8119-z
Received: 21 March 2008 / Revised: 17 September 2009
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Abstract This paper studies the autoregression models of order one, in a general time series setting that allows for weakly dependent innovations. Let \( \{X_t\} \) be a linear process defined by

\[
X_t = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k},
\]

where \( \{\psi_k, k \geq 0\} \) is a sequence of real numbers and \( \{\varepsilon_k, k = 0, \pm 1, \pm 2, \ldots\} \) is a sequence of random variables. Two results are proved in this paper. In the first result, assuming that \( \{\varepsilon_k, k \geq 1\} \) is a sequence of asymptotically linear negative quadrant dependent (ALNQD) random variables, the authors find the limiting distributions of the least squares estimator and the associated regression statistic. It is interesting that the limiting distributions are similar to the one found in earlier work under the assumption of i.i.d. innovations. In the second result the authors prove that the least squares estimator is not a strong consistency estimator of the autoregressive parameter \( \alpha \) when \( \{\varepsilon_k, k \geq 1\} \) is a sequence of negatively associated (NA) random variables, and \( \psi_0 = 1, \psi_k = 0, k \geq 1 \).

Key words ALNQD, autoregression models, least squares estimator, negatively associated, unit root test.

1 Introduction

Autoregression time series with a root have been the subject of much recent attention in the econometrics literature. This is partly because the unit root hypothesis is of considerable interest in applications, not only with data from financial and commodity markets where it has a long history but also with aggregate time series. At the theoretical level there has also been much recent research. This has concentrated on the distribution theory that is necessary to develop tests of the random walk hypothesis under null and the analysis of the power of various test under interesting alternatives. Investigations by Dickey\(^1\), Dickey and Fuller\(^2-3\), Phillips\(^4\), Wang, et al.\(^5\) have been at the forefront of this research. All of the research cited previously has been confined to the cases whereof innovations driving the model is independent of common variance or whereof the innovations from a martingale difference sequences. Therefore, it is important to develop tests for unit roots that do not depend on these conditions.

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\(^*\)This research is supported by the National Natural Science Foundation of China under Grant Nos. 10971081 and 11001104, and 985 Project of Jilin University.
\(^*\)This paper was recommended for publication by Editor Guohua ZOU.
Let \( \{y_t\}_{t=1}^{\infty} \) be a stochastic process generated according to
\[
y_t = \alpha y_{t-1} + X_t, \quad t = 1, 2, \ldots,
\]
where \( y_0 \) is a constant with probability one or has a certain specified distribution. Denote the ordinary least squares (OLS) estimator of \( \alpha \) by \( \hat{\alpha}_n = \frac{\sum_{t=1}^{n} y_t y_{t-1}}{\sum_{t=1}^{n} y_{t-1}^2} \). To test \( \alpha = 1 \) against \( \alpha < 1 \), a key step is to derive the limiting distribution of the well-known DF test statistic (Dickey and Fuller\(^2\))
\[
n(\hat{\alpha}_n - 1) = \frac{n^{-1} \sum_{t=1}^{n} y_{t-1} (y_t - y_{t-1})}{n^{-2} \sum_{t=1}^{n} y_{t-1}^2}.
\]

**Definition 1.1** A finite family of random variables \( \{\varepsilon_k, 1 \leq k \leq n\} \) is said to be negatively associated (NA) if for every pair of disjoint subsets \( A \) and \( B \) of the set of natural numbers \( \{1, 2, \ldots, n\} \),
\[
\text{Cov}\{f(\varepsilon_i, i \in A), \ g(\varepsilon_j, j \in B)\} \leq 0,
\]
where \( f \) and \( g \) are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated. This definition was introduced by Alam and Saxcean\(^6\). Joag-Dev and Proschan\(^7\) showed that many well-known multivariate distributions possess the NA property. Some examples include: a) The multinomial; b) The convolution of unlike multinomials; c) The multivariate hypergeometric; d) The Dirichlet; e) The Dirichlet compound multinomial; f) The negatively correlated normal distribution; g) The permutation distribution; h) The random sampling without replacement; and i) The joint distribution of ranks. Because of its wide application in multivariate statistical analysis and system reliability, the notation of NA has received considerable attention recently, the interesting reader is referred to [8].

Let \( \mathcal{C} \) be a class of functions of the form \( f(x_1, x_2, \ldots, x_j) : \mathbb{R}^j \mapsto \mathbb{R} \) \( (j \geq 1) \) which are coordinatewise monotonically nondecreasing. For two nonempty disjoint sets \( S, T \subset \mathbb{R} \), we define \( \text{dist}(S, T) \) to be inf \( \{|s - t|; s \in S, t \in T\} \). ALNQD was firstly defined by Zhang\(^8\).

**Definition 1.2** A random variables sequence \( \{\varepsilon_k, k \in \mathbb{Z}\} \) is said to be asymptotically linear negative quadrant dependent (ALNQD) if
\[
\rho^-(r) = \sup \{\rho^-(S,T); \text{dist}(S,T) \geq r, S,T \subset \mathbb{Z}\} \to 0 \quad \text{as} \quad r \to \infty.
\]
We define a quantity of dependence of \( X \) and \( Y \) by
\[
\rho^-(X,Y) = 0 \vee \sup \frac{\text{Cov}(f(X),g(Y))}{(\text{Var } f(X))^\frac{1}{2}(\text{Var } g(Y))^\frac{1}{2}},
\]
where the sup is taken over all \( f, g \in \mathcal{C} \) such that \( E|f(X)|^2 < \infty \) and \( E|g(X)|^2 < \infty \). For any disjoint subsets \( S, T \subset \mathbb{N} \), define
\[
\rho^-(S,T) = \sup \{\rho^-(X,Y); X \in F(S), Y \in F(T)\},
\]
where \( F(S) = \{ \sum_{k \in S} a_k \varepsilon_k; a_k \geq 0 \text{ and } a_k \neq 0 \text{ for finitely many } k's \} \), and \( F(T) \) is defined similarly.

It is obvious that the sequences of independent random variables, NA random variables, LNQD random variables (see [9]) and \( \rho^- \)-mixing random variables (see [9]) are all sequences of ALNQD random variables. In [9], the author obtained a CLT and an FCLT for the fields

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