Invariant holomorphic extension in several complex variables

Dedicated to Professor Sheng GONG on the occasion of his 75th birthday

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Abstract Two fundamental problems on the invariant holomorphic extensions have been posed, which are naturally arose from our solution of the extended future tube conjecture and closely and deeply related to the general theory of Stein manifolds due to Cartan-Serre. In this paper, the relationship is presented between the two problems, the motivation of considering the problems, and the methods to approach the problems. We have also posed some questions and conjectures related to this two problems.

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In ref. [17], a general problem about Steinness of complexification of invariant domains and a general approach to the problem have been presented, as a consequence, the extended future tube conjecture has been proved. Analytic continuation (or holomorphic extension) is a fundamental research subject in complex analysis. We will consider the invariant holomorphic extensions under group actions in the present paper, and we will first recall the most basic part of several complex variables and then present some relations between the general problem and some deep results in the general theory of several complex variables. Many further problems and conjectures are posed. It would lead to an interesting research topic. This paper is grown out of my plenary lecture at the International Conference on Several Complex Variables in honor of Prof. Gong Sheng in 2005.

It is well known that in one complex variable, given any domain $D \subset \mathbb{C}$, $\exists f \in \mathcal{O}(D)$ such that $D$ is the natural definition (existence) domain of the holomorphic function $f$ (i.e. $\partial D$ natural boundary), e.g. for the unit disc $\Delta \subset \mathbb{C}$, one can choose $f = \sum_{n=1}^{\infty} z^n$, or $\sum_{n=1}^{\infty} z^{2n}$.

We have similar equivalent definitions for holomorphic function in several complex variables as in one complex variable:
1. complex differential: \( f(a + h) = f(a) + l(h) + o(h) \) with \( l(h) \) being complex linear;
2. holomorphic: \( \bar{\partial}f = 0 \), i.e., satisfying Cauchy-Riemann condition;
3. complex analytic: \( f(z) = \sum_{v=(v_1,\ldots,v_n)\geq 0} a_v(z - a)^v, \ (z - a)^v := (z_1 - a_1)^{v_1} \cdots (z_n - a_n)^{v_n} \).

One of the most distinguished differences between several complex variables and one complex variable is given by the so-called Hartogs phenomenon: \( \exists \ D \subset \mathbb{C}^n \ (n \geq 2) \), s.t. \( \exists \ \tilde{D} \supset D \) and \( \forall f \in \mathcal{O}(D) \) one has \( f \in \mathcal{O}(\tilde{D}) \). e.g., the punctured unit ball \( D = B_n - \{0\} \) is such an example.

This causes one of the most fundamental concepts in several complex variables: a domain \( D \subset \mathbb{C}^n \) is called a domain of holomorphy if Hartogs phenomenon does not happen. (e.g. \( D = B_n - \{0\} \) is not a domain of holomorphy, while \( B_n \) is a domain of holomorphy. As pointed out above, any domain in complex plane is also a domain of holomorphy.)

One of the principal problems in several complex variables is to determine which domains are domains of holomorphy.

**Fact.** 1) A domain is a domain of holomorphy \( \iff \) it is holomorphically convex. (i.e., \( \exists f \in \mathcal{O}(D) \) s.t. \( \lim_{z \to \partial D} |f(z)| = \infty \).)

2) A domain is a domain of holomorphy \( \Leftrightarrow \) it is pseudoconvex. (i.e., \( \exists \varphi \in \text{psh}(D) \) s.t. \( \lim_{z \to \partial D} \varphi(z) = +\infty \).)

A function \( \varphi : D \rightarrow \mathbb{R} \cup \{-\infty\} \) is called plurisubharmonic (\( \varphi \in \text{psh}(D) \)) if a) \( \varphi \) is upper semi-continuous, b) \( \varphi|_{\{\text{complex line}\} \cap D} \) is subharmonic. Equivalently, \( i\partial\bar{\partial} \geq 0 \) in the sense of currents, e.g., Bergman kernel \( K(z,z) \) is plurisubharmonic. Given \( w \in \Omega \subset \mathbb{C}^n \), then \( \varphi(z) = K(z,w) \) minimizes functional \( \int_{\Omega} |\varphi(z)|^2 d\lambda(z) \) among all \( \varphi \in \mathcal{O}(\Omega) \cap L^2(\Omega) \) with \( \varphi(w) = K(w,w) \).

Define the action

\[
(S^1)^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n : (e^{i\theta_1},\ldots,e^{i\theta_n},(z_1,\ldots,z_n)) \mapsto (e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n).
\]

An invariant domain in \( \mathbb{C}^n \) w.r.t the action is called a Reinhardt domain. It is well known that

1) a Reinhardt domain \( D \subset (\mathbb{C}^*)^n \) is a domain of holomorphy \( \iff D \) is logarithmically convex, i.e., \( \log |D| := \{\log |z_1|,\ldots,\log |z_n|\} |z \in D \} \subset \mathbb{R}^n \) is convex.

2) a tube domain (an \( \mathbb{R}^n \)- invariant domain): \( T_\omega := \mathbb{R}^n + i\omega, \ \omega \subset \mathbb{R}^n \) is a domain of holomorphy \( \iff \omega \) is convex.

Analogue of domain of holomorphy in complex manifolds is Stein manifold. A complex manifold \( X \) is called a Stein manifold if it is i) holomorphically convex, ii) holomorphically separable. A complex manifold is Stein \( \iff \) it is a closed complex submanifold in some \( \mathbb{C}^N \). A Riemann surface is Stein \( \iff \) it is noncompact.

Solutions of the famous Levi problem, Cousin I, II problems etc. resulted in various important methods in several complex variables: - sheaf cohomology; - \( L^2 \) method; \( \cdot \cdot \cdot \) integral representation. Levi problem for complex manifold was solved by Grauert, as a corollary, it implies that any real analytic manifold can be real analytically embedded into \( \mathbb{R}^N \) for some large \( N \).