Three classes of smooth Banach submanifolds in $B(E,F)$

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Abstract

Let $E, F$ be two Banach spaces, and $B(E,F), \Phi(E,F), S\Phi(E,F)$ and $R(E,F)$ be the bounded linear, Fredholm, semi-Fredholm and finite rank operators from $E$ into $F$, respectively. In this paper, using the continuity characteristics of generalized inverses of operators under small perturbations, we prove the following result: Let $\Sigma$ be any one of the following sets: $\{T \in \Phi(E,F) : \text{Index} \ T = \text{const. and } \dim \ N(T) = \text{const.}\}$, $\{T \in S\Phi(E,F) : \text{either } \dim \ N(T) = \text{const. } < \infty \text{ or } \text{codim} \ R(T) = \text{const. } < \infty\}$ and $\{T \in R(E,F) : \text{Rank} \ T = \text{const. } < \infty\}$. Then $\Sigma$ is a smooth submanifold of $B(E,F)$ with the tangent space $T_A \Sigma = \{B \in B(E,F) : BN(A) \subset R(A)\}$ for any $A \in \Sigma$. The result is available for the further application to Thom’s famous results on the transversality and the study of the infinite dimensional geometry.

Keywords: semi-Fredholm operators, smooth submanifold, transversality, generalized inverse

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1 Introduction

Let $M$ and $N$ be $C^r$ ($r \geq 1$) Banach manifolds, $P$ a submanifold of $N$ and $f$ a $C^r$ ($r \geq 1$) map from $M$ into $N$. A famous result of Thom says that if $f \pitchfork P$ mod $N$, then the preimage $S = f^{-1}(P)$ is a submanifold of $N$ with the tangent space $T_x S = (T_x f)^{-1}(T_y P)$ for any $x \in S$, where $y = f(x)$ (see [1,2]). The result has been applied widely in differential topology, dynamic system and differential equations (see [3–7]). However, when applying this result, we are facing the rather difficult task for verifying the submanifold $P$ of $N$. For instance, let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial \Omega$, $E$ the Banach space $C^2_0(\Omega)$ of all functions with $u|_{\partial \Omega} = 0$ and the 2nd-order derivatives being $\alpha$-Lipschitz continuous and $F$ the Banach space $C^{0,\alpha}(\Omega)$ of all $\alpha$-Lipschitz continuous functions on $\Omega$. Cafagna[8] described the global invertibility of the mapping $F(u) = \Delta u + f(u)$ from $C^2_0(\Omega)$ into $C^{0,\alpha}(\Omega)$ and the finite solvability of the equation $F(u) = h$ in $C^{0,2}_0(\Omega)$. He presented the following result: the set $\Sigma$ of Fredholm operators of index $0$ with one dimensional kernel is a smooth hypersurface in $B(E,F)$ with a tangent space $T_A \Sigma = \{T \in B(E,F) : T N(A) \subset R(A)\}$ for any $A \in \Sigma$, where $N(\cdot)$ and $R(\cdot)$ denote the null space and the range of the underlying operator. But up to the present, we have not seen its proof yet. Thanks to it, the applications of the Thom’s result to the global invertibility of $F$ and the finite solvability of the equation $F(u) = h$ were made up.

In this paper, let $E$ and $F$ be the Banach spaces. Let $S\Phi(E,F), \Phi(E,F)$ and $R(E,F)$ be
the operators for semi-Fredholm, Fredholm and finite rank, respectively. We will prove the following results:

Let $\Sigma$ be any one of the following sets of the above operators:

- $\{T \in \Phi(E,F) : \text{Index } T = \text{const. and } \dim N(T) = \text{const.}\}$,
- $\{T \in S\Phi(E,F) : \text{either } \dim N(T) = \text{const.}< \infty \text{ or } \text{codim } R(T) = \text{const.}< \infty\}$,

and $\{T \in R(E,F) : \text{Rank } T = \text{const.}\}$. Then $\Sigma$ is a smooth submanifold of $B(E,F)$ with the tangent space $T_A\Sigma = \{B \in B(E,F) : BN(A) \subset R(A)\}$ for any $A \in \Sigma$.

In the point of view above one proves that these results are available for a further application to the Thom’s result and to the study of the infinite dimensional geometry.

In this section, we recall some concepts related to the continuity characteristics of generalized inverses of operators under small perturbations which are needed later.

Let $E$ and $F$ be the Banach spaces and $B(E,F)$ all of bounded linear operators from $E$ into $F$. Recall that an operator $A^+ \in B(F,E)$ is said to be a generalized inverse of $A \in B(E,F)$ provided $AA^+A = A$ and $A^+AA^+ = A$. Let GI($A$) denote all of the generalized inverses of $A$. It is well known that $A \in B(E,F)$ has a generalized inverse $A^+ \in B(F,E)$ if and only if $A$ is double splitting, i.e., $N(A)$ and $R(A)$ split $E$ and $F$, respectively. In the sequel we write all of double splitting operators in $B(E,F)$ as $B^+(E,F)$ and set $V(A,A^+) = \{T \in B(E,F) : \| (T - A)A^+ \| < 1\}$.

The following theorems on the perturbation of generalized inverses are known.

**Theorem 1.1**[6,7,9]. Suppose $A \in B^+(E,F)$. For any $A^+ \in \text{GI}(A)$, the following conditions are equivalent for any $T \in V(A,A^+)$:

- (i) $R(T) \cap N(A^+) = \{0\}$;
- (ii) $(I_E - A^+A)N(T) = N(A)$;
- (iii) $B = A^+C_{A^+}^{-1}(A^+,T) = D^{-1}(A^+,T)A^+$ is a generalized inverse of $T$, where $C_A(A^+,T) = I_F + (T - A)A^+$ and $D_A(A^+,T) = I_E + A^+(T - A)$.

Particularly, we have

**Theorem 1.2.** Let $A$ be a Fredholm operator, then $R(T) \cap N(A^+) = \{0\}$ for $T \in V(A,A^+)$ if and only if either $\dim N(A) = \dim N(T)$ or $\text{codim } R(T) = \text{codim } R(A)$.

**Theorem 1.3.** Let $A$ be a semi-Fredholm operator, then $R(T) \cap N(A^+) = \{0\}$ for $T \in V(A,A^+)$ if and only if either $\dim N(T) = \dim N(A) < \infty$ or $\text{codim } R(T) = \text{codim } R(A) < \infty$.

**Theorem 1.4.** Let $A$ be of finite rank, then $R(T) \cap N(A^+) = \{0\}$ for $T \in V(A,A^+)$ if and only if $\text{Rank } T = \text{Rank } A$.

For the details of the proofs see [7,9,10].

In the second section, a useful supplement to Theorem 1.1 and a split lemma of the space $B(E,F)$ are given. In this section, a class of locally $C^\infty$ diffeomorphisms are made up, which forms the local coordinate charts of smooth submanifolds $\Sigma$ as mentioned above.

### 2 Perturbation analysis of generalized inverses and a split lemma of $B(E,F)$

We first give some supplements to Theorem 1.1, which are closely related to the submanifolds $\Sigma$ above.

**Theorem 2.1.** Suppose that $A \in B^+(E,F)$ and $A^+ \in \text{GI}(A)$. Then the following conditions