On the inverse problem relative to dynamics of the \( w \) function

JIA ChaoHua

Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
(email: jiach@math.ac.cn)

Abstract

Let \( \mathcal{P} \) be the set of prime numbers and \( P(n) \) denote the largest prime factor of integer \( n > 1 \). Write

\[
C_3 = \{p_1 p_2 p_3 : p_i \in \mathcal{P} \ (i = 1, 2, 3), \ p_i \neq p_j \ (i \neq j)\},
\]

\[
B_3 = \{p_1 p_2 p_3 : p_i \in \mathcal{P} \ (i = 1, 2, 3), \ p_1 = p_2 \ or \ p_1 = p_3 \ or \ p_2 = p_3, \ but \ not \ p_1 = p_2 = p_3\}\.
\]

For \( n = p_1 p_2 p_3 \in C_3 \cup B_3 \), we define the \( w \) function by

\[
w(n) = P(p_1 + p_2)P(p_1 + p_3)P(p_2 + p_3).
\]

If there is \( m \in S \subset C_3 \cup B_3 \) such that \( w(m) = n \), then we call \( m \) \( S \)-parent of \( n \).

We shall prove that there are infinitely many elements of \( C_3 \) which have enough \( C_3 \)-parents and that there are infinitely many elements of \( B_3 \) which have enough \( C_3 \)-parents. We shall also prove that there are infinitely many elements of \( B_3 \) which have enough \( B_3 \)-parents.

Keywords: dynamics, prime number, \( w \) function

MSC(2000): 11N05

1 Introduction

In 2006, Goldring\[1\] proposed some problems and conjectures on dynamics of the \( w \) function and gave some interesting results. Recently Chen and Shi\[2,3\] made further progress on these problems. In this paper we shall study the inverse problem relative to dynamics of the \( w \) function.

We begin with some notations. Let \( \mathcal{P} \) be the set of prime numbers and \( P(n) \) denote the largest prime factor of integer \( n > 1 \). Write

\[
C_3 = \{p_1 p_2 p_3 : p_i \in \mathcal{P} \ (i = 1, 2, 3), \ p_i \neq p_j \ (i \neq j)\},
\]

\[
B_3 = \{p_1 p_2 p_3 : p_i \in \mathcal{P} \ (i = 1, 2, 3), \ p_1 = p_2 \ or \ p_1 = p_3 \ or \ p_2 = p_3, \ but \ not \ p_1 = p_2 = p_3\},
\]

\[
D_3 = \{p^3 : p \in \mathcal{P}\}.
\]

Then

\[
\{p_1 p_2 p_3 : p_i \in \mathcal{P} \ (i = 1, 2, 3)\} = C_3 \cup B_3 \cup D_3,
\]
where no any two of $C_3$, $B_3$ and $D_3$ intersect. Let

$$A_3 = C_3 \cup B_3.$$  

For $n = p_1p_2p_3 \in A_3$, define the $w$ function by

$$w(n) = P(p_1 + p_2)P(p_1 + p_3)P(p_2 + p_3)$$

and define

$$w^0(n) = n, \quad w^i(n) = w(w^{i-1}(n)), \quad i = 1, 2, \ldots,$$

which is reasonable according to Lemma 1 in Section 2.

Goldring\cite{1} proved that for any $n \in A_3$, there exists $i$ such that $w^i(n) = 20$. The smallest of such $i$ is denoted by $\text{ind}(n)$. Goldring\cite{1} also proposed two conjectures on $\text{ind}(n)$ (Conjectures 2.9 and 2.10) and gave the first upper bound for $\text{ind}(n)$ which has been improved greatly by Chen and Shi\cite{2} recently. Goldring\cite{1} also asked the following inverse problems:

1. For $n \in A_3$, can we find $m \in A_3$ such that $w(m) = n$?
2. If so, how many such elements are there?
3. What form do they have?

For $n \in A_3$, if there is $m \in A_3$ such that $w(m) = n$, then we call $m$ a parent of $n$. If this $m \in S \subset A_3$, then we call it $S$-parent of $n$. Goldring\cite{1} proved that there are infinitely many elements of $B_3$ which have at least seven parents. He proposed the following conjecture (Conjecture 2.16 in \cite{1}):

**Conjecture.** Every element of $A_3$ (respectively $B_3$) has infinitely many $C_3$-parents (respectively $B_3$-parents).

Chen and Shi\cite{3} proved that for any given positive integer $k$, there are infinitely many elements of $B_3$ which have at least $k$ $B_3$-parents. On the other hand, they\cite{3} proved that there are infinitely many elements of $B_3$ which have no $B_3$-parent.

In this paper we shall study parents of elements of $C_3$. It is obvious that the element of $C_3$ has no $B_3$-parent. We shall prove that there are infinitely many elements of $C_3$ which have enough $C_3$-parents.

In the following, $p$, $p_1$, $p_2$, $p_3$, $q$, $r$, $r_1$, $r_2$ denote prime numbers and $c_1$, $c_2$, $\ldots$ denote positive constants. The expression $f \ll g$ means $f = O(g)$. We suppose that $x$ is sufficiently large throughout.

**Theorem 1.** There exists an element $r_1r_2q$ of $C_3$ which satisfies $x^{\frac{1}{4}} \log x < r_1 < 2x^{\frac{1}{4}} \log x$ ($i = 1, 2$), $q \leq 4x$ and has at least $c_1 \frac{x}{\log^2 x}$ different $C_3$-parents $p_1p_2p_3$ with $x < p_i \leq 2x$ ($i = 1, 2, 3$).

We shall also prove that there are infinitely many elements of $B_3$ which have enough $C_3$-parents.

**Theorem 2.** There exists an element $qr^2$ of $B_3$ which satisfies $q \leq 4x$, $x^{\frac{1}{4}} \log x < r \leq 2x^{\frac{1}{4}} \log x$ and has at least $c_2 \frac{x}{\log^2 x}$ different $C_3$-parents $p_1p_2p_3$ with $x < p_i \leq 2x$ ($i = 1, 2, 3$).

Moreover, we shall prove that there are infinitely many elements of $B_3$ that have enough $B_3$-parents, being a quantitative improvement on the result of Chen and Shi\cite{3}.

**Theorem 3.** There exists an element $qr^2$ of $B_3$ which satisfies $x < q \leq 2x$, $x^{\frac{1}{4}} \log x < r \leq 2x^{\frac{1}{4}} \log x$ and has at least $c_3 \frac{x}{\log^2 x}$ different $B_3$-parents $pq^2$ with $x < p \leq 2x$. 