On interpolation for Hörmander’s algebras

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract We discuss some recent results on interpolation problems for weighted Hörmander’s algebras of holomorphic functions in several complex variables, and also give a sharp estimate on counting functions of interpolating varieties.

Keywords weight, pseudoconvex set, Hörmander’s algebra, interpolating variety, plurisubharmonic function, holomorphic function, entire function

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1 Introduction

Interpolation problems may be phrased in a very broad fashion. The one for Hörmander’s algebras $A_p(\Omega)$ of holomorphic functions $f$ in an open set $\Omega$ in $\mathbb{C}$ satisfying that

$$\sup_{z \in \Omega} \{|e^{-Ap(z)}|f(z)|\} < +\infty$$

for some $A > 0$, where $p$ is a plurisubharmonic weight function in $\Omega$ (cf. Section 2), is to study under what conditions an analytic variety $V$ in $\Omega$ is an interpolating variety for $A_p(\Omega)$ in the sense that every holomorphic function $f$ on $V$ satisfying an estimate of the form $|f(z)| \leq A e^{Bp(z)}$, $z \in V$, for some $A, B > 0$ always has an extension to a function in $A_p(\Omega)$. In this article, our attentions will be given to discrete varieties, which is the original setting for the classical interpolation theorem in one complex variable, going back to Weierstrass and Mittag-Leffler. In this situation, the problem is to find interpolation conditions so that there always exists a holomorphic function in the weighted space that has prescribed values, or more generally, prescribed finite collection of Taylor coefficients with growth conditions when multiplicities have to be taken into consideration.

Interests in this area arise from connections and applications of such questions to other subjects such as harmonic analysis, number theory, and systems theory (see e.g. [3–5, 9–13], etc.). When $\Omega$ is the unit disk in $\mathbb{C}$, interpolation conditions were studied by Carleson [14] and were used to give a necessary and sufficient condition for a finite set of functions to be generators for the ring $A_p(\Omega)$ when $p \equiv 0$. The analogous result on the generators for the ring $A_p(\Omega)$ was proved by Kelleher and Taylor [23] when $p(z) = |z|$ and $\Omega = \mathbb{C}$. This was then proved for general weights $p$ defined in an open (pseudoconvex) set of $\mathbb{C}^n$ by Hörmander [20]. It is well-known that interpolation for $A_p(\Omega)$, the idea theory for $A_p(\Omega)$, and
the study of systems of convolution equations are intrinsically related; and many questions in harmonic analysis, like finding all distribution solutions or finding out whether there are any to a system of linear partial differential equations with constant coefficients or, more generally, convolution equations in $\mathbb{R}^n$, can be translated into interpolation problems for $A_p$. For example, in harmonic analysis, the interpolation problem for the Hörmander algebras of entire functions in $\mathbb{C}^n$ is equivalent to some problems about solutions of the (overdetermined) system of convolution equations $u_i * f = g_i (1 \leq i \leq N)$, where $u_i$ (compactly supported distributions) and $g_i$ are known, and $f$ is the unknown (function). These questions arise in problems like robust filtering, imaging processing, etc. The $f$ represents an unknown signal, $u_i$ measuring devices and $g_i$ the output signals; i.e., data. The system can be solved when one could find deconvolvers $v_i$ such that $\sum v_i * u_i = \delta$ (deconvolution problem). The equivalent formulation of the deconvolution problem, via the Fourier transformations, is to find $v_i$ such that $\sum \hat{v}_i \hat{u}_i = 1$ (Bézout’s equation). By a theorem of Hörmander [20], the Bézout equation is solvable if there exist $\epsilon, C > 0$ such that $\sum |\hat{u}_j| \geq e^{-C p(z)}$ (strongly coprimeness). This is a Lojasiewich type inequality, which has a role in transcendental number theory. All these problems are closely related.

Note that interpolation problems for the algebras $A_p^0(\Omega)$ of holomorphic functions of minimal type satisfying that for every $\epsilon > 0$, $\sup_{z \in \Omega} |f(z)| e^{-\epsilon p(z)} < +\infty$ have been also studied due to connections to Dirichlet series, representation of solutions of infinite-order linear partial differential equations, etc., but will not be discussed in the present paper. We refer the reader to [3, 5, 8, 31, 46], and our recent work [25] for some related results and various references therein. We also refer the reader to the monographs [1, 36, 43], etc. for other interpolation problems for holomorphic and entire functions.

We will give some preliminaries for interpolation for weighted Hörmander’s algebras in the next section. In Section 3, we will discuss some of recent results on the subject with the main attention given to those by the author and his collaborators, and also give a sharp estimate on counting functions of interpolating varieties.

2 Preliminaries

We first recall the definitions of the weight $p$ and weight space $A_p(\Omega)$ (cf. [9, 20, 21], etc.). Let $\Omega$ be an open set in $\mathbb{C}^n$. A plurisubharmonic function $p : \Omega \to [0, \infty)$ is called a weight if it satisfies the following Hörmander conditions:

(i) all polynomials belong to $A_p(\Omega)$;

(ii) there exist constants $K_1, \ldots, K_4$ such that $z \in \Omega$ and $|\zeta - z| \leq e^{-K_1 p(z) - K_2}$ implies that $\zeta \in \Omega$ and $p(\zeta) \leq K_3 p(z) + K_4. \quad (2.1)$

Let $A(\Omega)$ be the ring of holomorphic functions in $\Omega$. The Hörmander algebra $A_p(\Omega)$ is defined as

$$A_p(\Omega) = \{ f \in A(\Omega) : \exists A, B > 0 \text{ such that } |f(z)| \leq Ae^{B p(z)}, z \in \Omega \}.$$ 

For examples and the meaning of the above conditions, the reader is referred to [20] and [9]. We note, however, that under the above conditions, $\Omega$ is necessarily pseudoconvex (see [20]), and $A_p(\Omega)$ is closed under differentiation.

The Hörmander algebras include those classically studied algebras of holomorphic and entire functions such as the algebra $A_{\mathcal{E}^t}(\mathbb{C}^n)$ of all entire functions of order $\leq \rho$ and finite type ($p(z) = |z|^\rho, \rho > 0$), the algebra $\mathcal{E}^t(\mathbb{R}^n)$ of Fourier transforms of distributions with compact support in $\mathbb{R}^n$ ($p(z) = |\text{Im} z| + \log(1 + |z|^2)$), and algebras of holomorphic functions in the unit ball $B_n$ such as $A^{-\infty}(B_n)$ ($p(z) = \log \frac{1}{1 - |z|}$). Many other Hörmander weights can be given. Note, however, that the conditions exclude the algebra of bounded holomorphic functions ($p(z) \equiv 0$).

Let $V = \{ \zeta_k \} \subset \Omega$ be a discrete set in $\Omega$. The weighted space $A_p(V)$ of sequences of complex numbers (described as “analytic” functions on $\Omega$) is defined as

$$A_p(V) = \{ \{ a_k \}_{k \in \mathbb{N}} : \exists A, B > 0 \text{ such that } |a_k| \leq Ae^{B p(\zeta_k)}, k \in \mathbb{N} \}.$$