Stability of multiquadric quasi-interpolation to approximate high order derivatives

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Abstract Numerical simulation of the high order derivatives based on the sampling data is an important and basic problem in numerical approximation, especially for solving the differential equations numerically. The classical method is the divided difference method. However, it has been shown strongly unstable in practice. Actually, it can only be used to simulate the lower order derivatives in applications. To simulate the high order derivatives, this paper suggests a new method using multiquadric quasi-interpolation. The stability of the multiquadric quasi-interpolation method is compared with the classical divided difference method. Moreover, some numerical examples are presented to confirm the theoretical results. Both theoretical results and numerical examples show that the multiquadric quasi-interpolation method is much stabler than the divided difference method. This property shows that multiquadric quasi-interpolation method is an efficient tool to construct an approximation of high order derivatives based on scattered sampling data even with noise.

Keywords numerical differential, radial basis functions (RBFs), Hardy’s multiquadric (MQ), quasi-interpolation, divided difference method, white noise, expectation, variance

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1 Introduction

This paper discusses a very basic problem in numerical approximation. Given a function \( f(x) \) defined on the interval \([a, b]\), one wants to simulate or estimate the high order derivatives \( f^{(n)}(x) \) of the function \( f(x) \) based on the sampling data \( \{f(x_j); \ j = 0, 1, \ldots, N\} \), where \( N + 1 \) is the number of data sites, and the data sites are in ascending order

\[ a = x_0 < x_1 < \cdots < x_N = b, \quad h := \max_{1 \leq j \leq N} (x_j - x_{j-1}). \]

Actually, the function values \( \{f(x_j); \ j = 0, 1, \ldots, N\} \) usually cannot be obtained exactly, because of the inaccuracy of measurement or the accuracy of computer. Assume that, we can only get a noised data \( \{f^*(x_j) = f(x_j) + \varepsilon_j\} \), where \( \{\varepsilon_j\} \) are pairwise independent white noise. For a random variable \( \varepsilon, E(\varepsilon) \)
denotes the expectation of \( \varepsilon \) and \( \sigma^2(\varepsilon) \) denotes its variance. The pairwise independent white noise \( \varepsilon_j \) should satisfy

\[
E\varepsilon_j = 0, \quad j = 0, \ldots, N
\]

and

\[
E\varepsilon_j\varepsilon_k = \begin{cases} 
\sigma^2, & j = k, \\
0, & j \neq k.
\end{cases}
\]

(1) means that the noise \( \{\varepsilon_j\} \) is unbiased.

The aim of this paper is to propose a novel numerical method to approximate the high order derivatives of \( f(x) \) based on the scattered sampling data \( \{f^*(x_j) = f(x_j) + \varepsilon_j\} \) with white noise \( \{\varepsilon_j\} \) defined as above.

The motivation to discuss this problem is as follows:

- The numerical calculation (estimation or simulation) of high order derivatives is very important in the numerical approximation, especially for the numerical solution of differential equations.
- There are only few methods to estimate the high order derivatives, except the divided differences in numerical approximation (e.g. [10, 11]).
- The noise will be divided by machine zero of the computer using the divided difference method as data sites become dense. That causes the instability of the divided differences. Therefore the divided differences can only be used to simulate lower order derivatives.
- A high accurate stable and simple method for simulating the high order derivatives is eagerly required in applications.

The divided differences are in fact to simulate the derivatives by using local polynomial interpolation, since the divided difference is the coefficient of the first term of the interpolatory polynomial [3]. Actually, if the sampling data contain noise, the interpolation method is inappropriate, because it treats noise as exact data, furthermore, polynomial interpolation possesses Runge’s phenomena. All in all, the divided difference method is strongly unstable. Unfortunately, one has not found other better methods except the divided differences to solve this problem until now. Using any other local interpolation methods to calculate the high order derivatives will cause the same problem too. In all noised situations, we would better relax the interpolation constraints. Moving least square method is perhaps a good candidate to take the place of divided difference method, however, we have not found any result for estimating high order derivatives by using moving least square.

Quasi-interpolation method is known as a simple and stable method to approximate functions themselves in approximation theory. The earliest case of quasi-interpolation is perhaps Bernstein’s approximation, which uses the Bernstein polynomials to construct a quasi-interpolation of a univariate function \( f(x) \) on \([0, 1]\). This scheme is widely used in computer aided geometric design under the names of Bezié and de Casteljau. Another well-known quasi-interpolation scheme is to reconstruct bandlimited functions via Whittaker-Shannon sampling series in signal processing. There is also the well-known B-spline series, which is included in many computer softwares for the representation of curves and surfaces. For the advantages of multiquadric function, multiquadric quasi-interpolation is also an useful approach in modern approximation theory and its applications [2], Beaton even used it in the Oscar award movie “The Lord of the Rings III” for computer animation. Define the multiquadric function \( \phi(x) = \sqrt{c^2 + x^2} \) and \( \phi_j(x) = \phi(x - x_j) \), where \( c \) is called the shape parameter. Multiquadric quasi-interpolation of a function on the scattered knots usually takes the form

\[
(Lf)(x) = \sum f(x_j)\psi_j(x),
\]

where \( \psi_j(x) \) are linear combinations of the multiquadric functions, i.e.,

\[
\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}.
\]

It is well-known that \( (Lf)(x) \) converges to \( f(x) \) (see [4, 12]). Can this method be used to approximate the derivatives of the functions? Ma and Wu [8] showed that multiquadric quasi-interpolation method