Extremal polygonal quasiconformal mappings and biLipschitz mappings

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Abstract We show that the extremal polygonal quasiconformal mappings are biLipschitz with respect to the hyperbolic metric in the unit disk.

Keywords quasiconformal mapping, extremal polygonal quasiconformal mapping, hyperbolic metric

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1 Introduction

Let $D = \{z : |z| < 1\}$ and $\partial D = \{z : |z| = 1\}$. An orientation-preserving homeomorphism $h : \partial D \to \partial D$ is said to be quasisymmetric if there is a quasiconformal mapping $f : D \to D$ such that $f|_{\partial D} = h$.

Assume that $f : D \to D$ is a quasiconformal mapping. The complex dilatation (or Beltrami coefficient) of $f$ is denoted by $\mu_f$ and the maximal dilatation is denoted by $K(f) = \sup_{z \in D} K_z(f)$, where $K_z(f) = 1 + |\mu_f(z)| / (1 - |\mu_f(z)|), z \in D$.

Let $QS$ be the space of all quasisymmetric homeomorphism of $\partial D$ and let $QC(D)$ be the space of all quasiconformal homeomorphism of $D$. For $h \in QS$, set $[h] = \{f \in QC(D) : f|_{\partial D} = h\}$ and $K[h] = \inf_{f \in [h]} K(f)$. By the normal family argument, there always exists at least one quasiconformal mapping $f_0$ in $[h]$, such that $K(f_0) = K[h]$. The quasiconformal mappings $f \in [h]$ such that $K(f) = K[h]$ are called extremal quasiconformal mappings in $[h]$. If there is a unique quasiconformal mapping $f_0$ such that $K(f_0) = K[h]$, then we call $f_0$ uniquely extremal. If the complex Beltrami coefficient $\mu_{f_0}$ is of form $k \frac{\varphi}{|\varphi|}$, where $k = \frac{K(f_0)-1}{K(f_0)+1}$, and $\varphi$ is a quadratic holomorphic differential with finite norm

$$\|\varphi\| = \int_D |\varphi| \, dx \, dy,$$

where $z = x + iy$, then we call $f_0$ a Teichmüller mapping and $\varphi$ the associated quadratic differential for $f$. It is well known that every Teichmüller mapping is unique extremal in its homotopic class.

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In this paper, we shall be mainly concerning the relationship between extremal quasiconformal mappings and biLipschitz mappings. Now we recall the definition of the biLipschitz mappings.

**Definition.** Let $f$ be a homeomorphism from hyperbolic surface $S$ to another hyperbolic surface $R$, if there exists a constant $B$ such that

$$B^{-1}d_S(z,w) \leq d_R(f(z), f(w)) \leq Bd_S(z,w)$$

for all $z, w \in S$, where $d_S, d_R$ are the hyperbolic metrics of $S, R$ respectively, then we call $f$ biLipschitz with respect to the hyperbolic metrics with Lipschitz constant $B$.

In the following, our model for the hyperbolic surface will be the unit disk or the upper half plane in $\mathbb{C}$. A map $f$ is said to be biLipschitz if $f$ is a self homeomorphism of the unit disk or the upper half plane in $\mathbb{C}$ and is biLipschitz with respect to the hyperbolic metric of the unit disk or the upper half plane in $\mathbb{C}$.

The first known quasiconformal mappings which are also biLipschitz are the Beurling-Ahlfors extensions of the quasisymmetric homeomorphisms of the real axis [1]. Later, one knows that the Douady-Earle extensions to unit disk of quasisymmetric homeomorphisms of unit circle are also biLipschitz [3].

Recently, motivated from the studies of Sullivan hyperbolic convex hull theorem for simply connected domains in the plane [4], Bishop obtained the following interesting result.

**Theorem A** [2]. Given $K < \infty$ and $\epsilon > 0$, there is a $C < \infty$ so that if $f$ is $K$-quasiconformal map of the upper half plane $\mathbb{H}$ to itself, then there is a $(K + \epsilon)$-quasiconformal map $g : \mathbb{H} \to \mathbb{H}$ which is $C$-biLipschitz with respect to the hyperbolic metric on $\mathbb{H}$ and which agrees with $f$ on $R = \partial \mathbb{H}$.

As Bishop pointed out in [2] that not every extremal quasiconformal mapping is biLipschitz and in this sense, Theorem A is sharp. So far, we do not know much about what kinds of extremal quasiconformal mappings are biLipschitz, even about uniquely extremal quasiconformal mappings. Thus it might be interesting to find some class of quasiconformal mappings such that they are both extremal and biLipschitz. In this paper, we shall consider this problem and find that the extremal polygonal quasiconformal mappings are in this class.

Polygonal quasiconformal mappings are introduced by Strebel (cf. [6] and [8]) and they play some fundamental roles in the theory of extremal quasiconformal mappings. Now we introduce their definition. The unit disk $D$ with $n \geq 4$ anticlockwise ordered distinguished points $z_1, z_2, \ldots, z_n$ on $\partial D$, which denoted by $D(z_1, z_2, \ldots, z_n)$, is called an $n$-polygon. For a pair of $n$-polygon $D(z_1, z_2, \ldots, z_n)$ and $D(w_1, w_2, \ldots, w_n)$ with vertices corresponding to each other in the same order, there always exists a uniquely extremal Teichmüller mapping $f : D(z_1, z_2, \ldots, z_n) \to D(w_1, w_2, \ldots, w_n)$ such that $f(z_j) = w_j$, for $j = 1, 2, \ldots, n$ (cf. [6] and [8]). The Teichmüller mapping $f$ will be called an extremal $n$-polygonal quasiconformal mapping or extremal polygonal quasiconformal mapping and its associated quadratic differential is called an $n$-polygonal or a polygonal differential. It is known that every extremal quasiconformal mapping can be approximated by a sequence of extremal polygonal quasiconformal mappings $\{f_n\}$. For every extremal quasiconformal mapping, Reich and Strebel proved that there is a Hamilton sequence formed by polygonal differentials. Refer to [6] and [8] for more information about this subject.

In this paper, we shall prove the following result.

**Theorem 1.1.** For $n \geq 4$, let $f : D(z_1, z_2, \ldots, z_n) \to D(w_1, w_2, \ldots, w_n)$ be the extremal $n$-polygonal quasiconformal mapping. Then $f$ is biLipschitz with respect to the hyperbolic metric in $D$.

## 2 Proof of Theorem 1.1

The proof of Theorem 1.1 uses some basic geometric properties of polygonal quasiconformal mappings. It is well known that for a Teichmüller mapping $f$, there always exist two quadratic differentials associating it. For an $n$-polygonal extremal mapping, the associated quadratic differential $\phi$ and $\psi$ respectively, are $n$-polygonal differentials. The two differentials are real along the sides of the boundary of the polygons.