An equivalence canonical form of a matrix triplet over an arbitrary division ring with applications

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Abstract We in this paper give a decomposition concerning the general matrix triplet over an arbitrary division ring $F$ with the same row or column numbers. We also design a practical algorithm for the decomposition of the matrix triplet. As applications, we present necessary and sufficient conditions for the existence of the general solutions to the system of matrix equations

$$DXA = C_1, \quad EXB = C_2, \quad FXC = C_3$$

and the matrix equation

$$AXD + BYE + CZF = G$$

over $F$. We give the expressions of the general solutions to the system and the matrix equation when the solvability conditions are satisfied. Moreover, we present numerical examples to illustrate the results of this paper. We also mention the other applications of the equivalence canonical form, for instance, for the compression of color images.

Keywords division ring, linear matrix equation, equivalence canonical form of a matrix triplet, decomposition

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1 Introduction

Throughout this paper, we denote an arbitrary division ring by $F$, the set of all $m \times n$ matrices over $F$ by $F^{m \times n}$, the set of all $n \times n$ invertible matrices over $F$ by $GL_n(F)$. The symbols $I_i$, $C(A)$, $R(A)$, $\dim R(A)$ stand for the $i \times i$ identity matrix, the column right space, the row left space of a matrix $A$ over $F$, the dimension of $R(A)$, respectively. By [16], for a matrix $A$ over $F$, $\dim R(A) = \dim C(A)$, which is called the rank of $A$ and denoted by $r(A)$. Moreover, we denote the set of all $m \times n$ matrices over $F$ with rank $r$ by $F^{m \times n}_r$.

Matrix theory over division rings is an important part of division ring theory. Many mathematicians, such as Cohn [3], Guralnick [11,12], Gustafson [14], Wiegmann [37], Lam and Leroy [17], Hartwig and

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Putcha [15], Šemrl [25], etc., have made excellent contributions to matrix theory over $\mathcal{F}$. Since, however, the commutative law of multiplication does not hold over $\mathcal{F}$, the results of matrix theory over $\mathcal{F}$ have not been more fruitful so far than those over fields, in particular, decompositions of matrices and matrix equations.

For matrix factorization, there are many kinds of decompositions, for instance, the generalized singular decompose [19, 22], the decomposition of triple matrices which two of them have the same row numbers meanwhile two of them have the same column numbers [2, 13], and so on. To our knowledge, there has been little information on the equivalence decomposition of three matrices with the same row or column numbers.

We know that matrix equation is one of the topics of very active research in matrix theory and applications, and a large number of papers have presented several methods for solving several matrix equations (e.g. [4, 6–10, 18, 30–36, 40, 42]). In mathematics, engineering, and others, many problems can be transformed into some linear matrix equations. For example, the solvability of the almost and the exact noninteracting control problems without internal stability has been shown in [20, 39] to be equivalent to the existence of a solution to the system of matrix equations

$$A_1X B_1 = C_1, \quad A_2X B_2 = C_2$$

over some rings, which was widely investigated (see, e.g., [1, 5, 20, 21, 26, 41]). The growth curve model in statistics is consistent if and only if the matrix equation

$$AXB + CYD = E$$

is consistent [28]. A regression model related to (2) is

$$M = AXB + CYD + \varepsilon,$$

where both $X$ and $Y$ are unknown parameter matrices and $\varepsilon$ is a random error matrix. This model is also called the nested growth curve model in the literature (see, e.g. [24, 27]). As an extension of (1) and (2), the following system of matrix equations

$$DXA = C_1, \quad EXB = C_2, \quad FXC = C_3$$

and the matrix equation

$$AXD + BYE + CZF = G$$

over a field were investigated by Tian in [29]. The solvability conditions to (3) and (4) were given. However, to our knowledge, there has been little information on the general solutions of (3) and (4) so far.

Motivated by the mentioned above, we concentrate on establishing an equivalence canonical form of triple matrices with the same row or column numbers over $\mathcal{F}$, and use the decomposition to investigate the matrix equations (3) and (4) over $\mathcal{F}$.

The remainder of the paper is organized as follows. In Section 2, we first propose the equivalence canonical form of a general matrix triplet with the same row numbers and one with the same column numbers over $\mathcal{F}$, that is, we are going to proceed the simultaneous decomposition of three matrices with the same row (or column) numbers over $\mathcal{F}$. The following we intend to design a practical algorithm for the decomposition of the matrix triple. Some special cases of the decomposition are also considered. In Section 3, using the equivalence canonical form, we derive necessary and sufficient conditions for the existence of the general solutions to (3) and (4) over $\mathcal{F}$. The expressions of the general solutions to the system and the equation are also given when the solvability conditions are met. In Section 4, we present two numerical examples to illustrate the results of this paper. In closing this paper, we in Section 5 give a conclusion and propose further research topics related to the equivalence canonical form.