Symmetry and nonexistence of positive solutions to an integral system with weighted functions

DOU JingBo1,2, QU ChangZheng1,* & HAN YaZhou3

1Center for Nonlinear Studies, Northwest University, Xi’an 710069, China;
2School of Statistics, Xi’an University of Finance and Economics, Xi’an 710100, China;
3Department of Mathematics, College of Science, China Jiliang University, Hangzhou 310018, China

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Abstract Consider the system of integral equations with weighted functions in \( \mathbb{R}^n \),

\[
\begin{align*}
    u(x) &= \int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^n} dy, \\
v(x) &= \int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^n} dy,
\end{align*}
\]

where \( 0 < \alpha < n \), \( \frac{1}{p} + \frac{1}{q} > \frac{n-\alpha}{n} \), \( \frac{\alpha}{n} < p, q < \infty \), \( Q(x) \) and \( K(x) \) satisfy some suitable conditions. It is shown that every positive regular solution \((u(x), v(x))\) is symmetric about some plane by developing the moving plane method in an integral form. Moreover, regularity of the solution is studied. Finally, the nonexistence of positive solutions to the system in the case \( 0 < p, q < \frac{n+\alpha}{n-\alpha} \) is also discussed.

Keywords Hardy-Littlewood-Sobolev inequality, system of integral equations, symmetry, regularity, conformally invariant property

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1 Introduction

Let \( 0 < \alpha < n \) and \( r_0, s_0 > 1 \) such that \( \frac{1}{r_0} + \frac{1}{s_0} = \frac{\alpha}{n} \). The classical Hardy-Littlewood-Sobolev (HLS) inequality states that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^\alpha g(x) f(y) dx dy \leq C(n, s_0, \alpha) \|f\|_{L^{r_0}} \|g\|_{L^{s_0}}
\]

for \( f \in L^{r_0}(\mathbb{R}^n) \), \( g \in L^{s_0}(\mathbb{R}^n) \). It is noted that the maximizer \( f, g \) of the best constant \( C(n, s_0, \alpha) \) in (1) relates closely to the following system in \( \mathbb{R}^n \),

\[
\begin{align*}
    u(x) &= \int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^n} dy, \\
v(x) &= \int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^n} dy,
\end{align*}
\]
where \( \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\alpha}{n}, \frac{\alpha}{n-\alpha} < p, q < \infty \). Under the natural integrability conditions \( u \in L^{p+1}(\mathbb{R}^n) \) and \( v \in L^{q+1}(\mathbb{R}^n) \) with \( p, q \geq 1 \), Chen et al. [2, 4] discussed the symmetric and monotonic properties of the solutions to system (2) by using an integral form of the method of moving planes, and they also proved the regularity of the solutions by the contracting mapping theorem. Recently, under the weaker assumptions, Hang [10] proved the symmetry, monotonicity and smoothness of the solutions \((u, v)\) to system (2), i.e., if \( u \in L^{p+1}(\mathbb{R}^n) \) is nonnegative for \( \frac{n-\alpha}{n} < p, q < \infty \) and does not vanish identically, then \( u, v \in C^\infty \) and \( u, v \) are radially symmetric with respect to some point \( x_0 \in \mathbb{R}^n \) and strictly decreasing along radial direction.

It is worthy of mentioning that system (2) is closely related to the system of PDEs

\[
\begin{align*}
(-\Delta)^\frac{\alpha}{n} u &= v^q, \quad u > 0, \quad \text{in} \ \mathbb{R}^n, \\
(-\Delta)^\frac{\alpha}{n} v &= u^p, \quad v > 0, \quad \text{in} \ \mathbb{R}^n.
\end{align*}
\]  

(3)

The system (3) arises from the higher order semilinear Schrödinger type systems in \( \mathbb{R}^n \). Recently, for \( \alpha = 2m, m \in \mathbb{N}_+ \) and \( 2m < n \), Guo et al. [9] and Zhang [15] proved the following Liouville Theorem:

(i) If \( p, q \geq 1 \), but not both equal to 1 such that

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2m}{n},
\]

then system (3) has no radially positive classical solutions.

(ii) If \( p, q \geq 1 \) but not both equal to 1, satisfying

\[
(n-2m)q < \frac{n}{p} + 2m, \quad \text{or} \quad (n-2m)p < \frac{n}{q} + 2m,
\]

or

\[
\frac{2m(p+1)}{pq-1}, \quad \frac{2m(q+1)}{pq-1} \in \left[ \frac{n-2}{2}, n-2m \right),
\]

then there exist no positive classical solutions to system (3).

(iii) If \( p \) and \( q \) satisfy \( 1 \leq p, q \leq \frac{n+2m}{n-2m} \), not equal to 1 or \( \frac{n+2m}{n-2m} \) simultaneously, then system (3) has no positive classical solutions. If \( p = q = \frac{n+2m}{n-2m} \) then the positive classical solutions of system (3) are radially symmetric with respect to some point of \( \mathbb{R}^n \).

It is noted that the case \( \alpha \neq 2m \) is not mentioned in [9,15] and the case \( 0 < p, q < 1 \) is not mentioned in [2, 4, 9, 15]. In this paper, we pay more attention to the two cases. Indeed, we study the following more general system of integral equations in \( \mathbb{R}^n \):

\[
\begin{align*}
u(x) &= \int_{\mathbb{R}^n} \frac{Q(y)|v(y)|^q}{|x-y|^{n-\alpha}} dy, \\
v(x) &= \int_{\mathbb{R}^n} \frac{K(y)|u(y)|^p}{|x-y|^{n-\alpha}} dy.
\end{align*}
\]  

(4)

where \( 0 < \alpha < n, n \geq 1 \), \( Q(x) \) and \( K(x) \) are positive functions satisfying the following assumptions (A1) and (A2).

(A1) \( Q(x), K(x) \) are symmetric in \( x_1 \) and are monotone decreasing for \( x_1 \geq 0 \).

(A2) \( Q(x)|v(x)|^{q-r^{-1}q_1} \) and \( K(x)|u(x)|^{p-r^{-1}q_2} \) are integrable on any domain which is of a positive distance away from the plane \( x_1 = 0 \), where \( q_1 = \frac{q+1}{q-r}, \ q_2 = \frac{p+1}{p-r} \), \( 1 \leq r < p \) and \( \frac{1}{r} \leq \frac{1}{q} \).

Interestingly, Chen et al. [5] discussed \( L^\infty \) regularity, symmetry and monotonicity of the single integral equation with a weighted function

\[
u(x) = \int_{\mathbb{R}^n} \frac{K(y)|v(y)|^p}{|x-y|^{n-\alpha}} dy.
\]

We first have the following result.